Kaplansky's Conjectures and Actions on CAT(-1) Spaces

by

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Previous Degrees
Bachelor of Science, Arizona State University, 2020

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

In the Graduate Academic Unit of Mathematics and Statistics

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This thesis is accepted by the
Dean of Graduate Studies

THE UNIVERSITY OF NEW BRUNSWICK

November 2023

Abstract

We provide specific conditions on a ring R and a group G under which the group ring RG will satisfy the Kaplansky Conjectures on the existence of non-trivial units, zero-divisors and idempotents in the group ring. We give a chain of implications on properties that a group must have to satisfy these conjectures. Specifically, we define a Bowditch action of a group on a type of metric space called a CAT(-1) space and show this action will be spherically diffuse. We then prove that if a group acts on a metric space in a spherically diffusely, then the group itself must be diffuse. Next we prove that if a group is diffuse then it satisfies the Unique Product Property. We then prove that if a group satisfies this property, then the group ring formed by this group and any integral domain will satisfy the Kaplansky Conjectures.

Dedication

I would like to dedicate this to my family and friends who never doubted me. Mom, Dad, Nick, Jackie, Katie, Bailey, Garrett, Liam, Maddie, and people I may be forgetting. You believed in me when I didn't believe in myself. There are not enough words in the English language to thank you. I love you all.

Acknowledgements

First, I need to acknowledge and thank my supervisor Dr. Nicholas Touikan. Despite my numerous dumb questions, he stood by my side and helped me learn so much. He gave me a chance and none of this would have been possible without his tutelage. I would also like to thank my supervisory committee that consisted of Dr. Touikan as well as Dr. Vlad Tasić and Dr. Branimir Ćaćić. They provided constructive criticism and helped this thesis see the light of day.

I would also like to thank my various university professors over the years. Dr. Touikan and Dr. Ćaćić taught me Geometric Group Theory and Algebraic Topology, respectively. I would also like to thank Dr. Sara Faridi and Dr. Andrea Fraser, faculty members from Dalhousie University. Dr. Faridi taught me Field and Galois Theory and Module Theory, while Dr. Fraser taught me Measure Theory. These professors of mine that I have taken classes with over my masters program have shaped me and have introduced me to new flavors of mathematics that I have never seen before.

I'll also thank my various undergraduate math professors from Arizona State University. Dr. Samantha Dahlberg, Dr. Richard Reynolds, Dr. Susanna Fishel, Dr. John Jones, Dr. Konrad Aguilar, Dr. Marko Samara, Dr. Brett Kotschwar, Dr. Richard Ruedemann, Dr. Terri Miller, Dr. Benjamin Carrillo and Mrs. Jennifer Boney. All of these professors of mine have shaped my experiences, from classes in Calculus I to

Enumerative Combinatorics to Ring Theory. A special thanks goes to Dr. Andrzej Czygrinow, my undergraduate thesis supervisor. He helped me write my first paper in higher level mathematics, and I have grown so much from those experiences.

Thanks too goes to every math teacher. Mrs. Mahony, Mr. Miles, Mr. Pearson, Mrs. Leonard, Mrs. Hoke, Mrs. Corral and Mr. Weis - you have all shaped me over the years. I vividly remember struggling with long division in fifth grade - now I am writing my master's thesis in abstract algebra, metric space theory and hyperbolic geometry. I have come so far and have you all to thank.

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Vita

Chapter 1

Groups, Rings, Group Rings and the Kaplansky Conjectures

Perhaps unsurprisingly, a group ring is, informally, a ring that is constructed from a ring and a group. Less informally, it is a ring that is at the same time a free module. For a ring R and a group G, the group ring, which we denote by RG, is a free R-module with basis G. In practice, this means that elements of a group ring consist of finite linear combinations of group elements with coefficients coming from the ring; for example,

$$r_1g_1 + r_2g_2 + \cdots + r_ng_n$$

for each $r_i \in R$ and each $g_i \in G$. An equivalent way of thinking about the group ring RG is that of functions from G to R with finite support - i.e., only finitely many outputs of the function $f\colon G \to R$ are nonzero. Under this construction, multiplication will be done by means of convolution, a non-standard way of multiplying functions so as to mimic the "foiling" multiplication of standard group ring elements. We will explore this relationship in depth in Section 1.2.

Despite their well-studied building blocks, there are many unanswered questions on group rings. In 1956, Irving Kaplansky made several ring-theoretical conjectures in order to "...help rekindle interest in the theory of rings." [Kap70]. Many of these conjectures have been answered over the years, but his conjectures on group rings remained elusive until only recently. As a logistical aside, we mention that even though there are twelve original conjectures from Kaplansky's 1956 proposal, we will informally be referring to the most famous, and until recently unproven, trio of conjectures as the "Kaplansky Conjectures". This trio consists of the following question:

> If R is a ring and G is a torsion-free group, then when does RG lack non-trivial units, non-trivial zero-divisors and non-trivial idempotents?

We explore the specific requirements - for example, why torsion even matters in this context - as well as what we mean by "non-trivial" in Sections 1.3 and 1.4. Furthermore, these conjectures are interrelated, and there is a chain of implications that links them. It's easy to see that if a ring is an integral domain, then it lacks non-trivial idempotents, and this in turn means the the Zero Divisor Conjecture implies the Idempotent Conjecture. The fact that the Unit Conjecture implies the Zero Divisor Conjecture is not as obvious, but Connell's paper [Con63] proved this. We will explore these relationships later in Chapter 1.

There are have been many different approaches so far as to tackle these conjectures. We mention that the circle of ideas related to these approaches is standard folklore by now, and we will be proving these claims carefully in the thesis, but we mean to emphasize that these are not original contributions by the author. The most well-known avenue is to assume the base group of a group ring satisfies a property known as the *unique product property.* If it does, then the Kaplansky conjectures are upheld, and the group ring will lack non-trivial units, zero-divisors idempotents. The unique product property is upheld by a group when, for every finite subset S

of the group G, there exists $g \in G$ such that there exist unique $x_g, y_g \in S$ such that $x_g y_g = g$. A similar notion of the *two unique product property* also exists, and as the name suggests, it means that a group element can be written in two ways as a product. Strojnowski shows in [Str80] that, despite their names, the unique product property and the two unique product property are actually equivalent. We use Chapter 2 to explore these unique product properties and how they relate to the Kaplansky conjectures.

Using this avenue, in [Bow00], Bowditch discussed geometric properties of certain groups and their relationship to the Kaplansky conjectures. Bowditch introduces the notion of *diffuseness*, and the related notion of *weak diffuseness*. Broadly speaking, these notions refer to "extreme elements" of a subset of a group, those that are "on the border" of a certain set and are "pushed out" by the action of a group element. Bowditch, in the paper, proves that diffuse groups satisfy the unique product properties, and therefore the Kaplansky conjectures are upheld by a group ring whose base group is diffuse. Linnell and Witte Morris in [LWM14] show then that Bowditch's notions of diffuseness and weak diffuseness are equivalent. We explore these notions more in Chapter 3.

Delzant, in his paper [Del97], builds off of a result given by Gromov in [Gro87]. Gromov had previously shown that a group acting by isometries on a CAT(-1) space - a space we define carefully in Chapter 4 and, for now, liken to a sort of hyperbolic space - coupled with a lower bound on translation length of $4\ln(3)$ will imply the Kaplansky conjectures. Delzant, in his paper [Del97], deduces the Kaplansky conjectures in hyperbolic groups with high translation length. Bowditch, in his paper [Bow00], also builds off of Gromov, and improves this bound on the translation length to $2\ln(1+\sqrt{2})$.

We are inspired by both of these papers, and it is from this inspiration that the thesis is born. We take a condition considered by Bowditch and craft a definition out of it - we define a type of action of a group on a metric space called *spherical diffuseness*. Given a group G that acts on a metric space (X, d), we will say the action is spherically diffuse if

$$d(x, y) < \max\{d(x, g \cdot y), d(x, g^{-1} \cdot y)\}$$

for every $x, y \in X$ and every non-trivial $g \in G$. Bowditch considered the above inequality and showed that what we call spherical diffuseness implies weak diffuseness, and therefore diffuseness.

We take this fleshed out definition and then show that if a group acts on a negatively curved metric space - specifically, a CAT(-1) space - with a sufficiently high translation length, then the action is spherically diffuse. This will allow us to follow the chain of implications down to the Kaplansky conjectures. We explore this more in Chapter 4, and emphasize that the content at the end of Chapter 4 - specifically Theorems 4.4.2 and 4.5.1 - are original contributions by the author.

Everything we have mentioned thus far discusses various ways to *affirm* Kaplansky's conjectures; this is primarily because a counterexample did not exist for over sixty years. Indeed, it was only in 2021 that Giles Gardam, in [Gar21], found a counterexample to the unit conjecture. The ring he used was a field of characteristic 2, and the group he used was a type of crystallographic group.

While our work does not rise to the level of showing a counterexample, we provide a sufficient condition on a group acting on a negatively curved metric space under which these ring theoretical conjectures are upheld.

Let's start off by refreshing our memory on key definitions.

1.1 Groups and Rings

We use this introductory section to recall some basic definitions, taken from the classic algebra reference by Dummit and Foote. [DF04]

Definition 1.1.1. A *group* $(G, \star)$ is a set G with a binary operation $\star$ that satisfies the following axioms:

1. the binary operation is associative
2. the group has an identity element
3. the group is closed under inverses

Definition 1.1.2. A *ring* $(R, +, \times)$ is a set R equipped with two binary operations $+$ and $\times$ - respectively called addition and multiplication - that satisfies the following axioms:

1. associative addition
2. additive identity element; we refer to this element as *zero* and denote it by 0_R, or just 0 is the specific ring R is clear from context
3. additive inverses
4. commutative addition; as an aside, we mention that these first four axioms make R an abelian group under addition
5. associative multiplication
6. distributivity of multiplication over addition - for every $r, s, t \in R$, we have not only that $r \times (s + t) = (r \times s) + (r \times t)$, but that $(r + s) \times t = (r \times t) + (s \times t)$

A ring is said to be *commutative* if it has commutative multiplication. A ring is said to be *with identity* if there exists a multiplicative identity element. When it exists, we refer to this element as *one* and denote it by 1_R, or just 1 is the specific ring R

is clear from context. If a ring is both commutative and with identity, we refer to it as a *commutative ring with identity.*

Throughout the thesis we will encounter two special types of rings, which we define here.

Definition 1.1.3. A commutative ring with identity R is called an *integral domain* if the equation $a \times b = 0_R$ implies that $a = 0_R$ or $b = 0_R$.

Definition 1.1.4. A commutative ring with identity F is called a *field* if for every non-zero $x \in F$ has a multiplicative inverse.

1.2 Group Rings

As is the case with so many stories in math, we try to create a new object from two existing ones - in this case a group and a ring. It is from here that the group ring is born.

For a group G and a ring R with identity 1_R, we define the set RG to be

$$RG := \left\{ \sum_{g \in G} r_g g \mid r_g \in R \text{ and all but finitely many } r_g = 0_R \right\}.$$

We may identically define this set by

$$RG := \left\{ \sum_{g \in I} r_g g \mid r_g \in R, I \subseteq G \text{ and } |I| < \infty \right\}.$$

We equip this set with the binary operations of addition and multiplication as follows. Define an additive identity on RG by $0_{RG} := \sum_{g \in G} 0_R g$. We then define addition to be given by

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g := \sum_{g\in G} (r_g +_R s_g) g.$$

where $+_R$ is the ring addition from R. We define multiplication to be given by

$$\left(\sum_{g\in G} r_g g\right) \times \left(\sum_{g\in G} s_g g\right) := \sum_{g\in G} \left(\sum_{h\in G} r_h s_{h^{-1}g}\right) g.$$

Note that RG is closed under addition and multiplication as defined above. That RG is closed under addition is obvious. As for multiplication, let's use the second definition of RG to see why this is true. Assume $\sum_{g\in G} r_g g = \sum_{g\in I_1} r_g g$ and that $\sum_{g\in G} s_g g = \sum_{g\in I_2} s_g g$ for some finite subsets $I_1, I_2 \subseteq G$. Upon examining $\sum_{g\in G} \left(\sum_{h\in G} r_h s_{h^{-1}g}\right) g$, we note that even in the worst case scenario in which every single r_h and $s_{h^{-1}g}$ is nonzero for $h \in I_1$ and $h^{-1}g \in I_2$ and where $r_h s_{h^{-1}g} \neq 0_R$ (in the case where the ring is not an integral domain), then $\sum_{g\in G} \left(\sum_{h\in G} r_h s_{h^{-1}g}\right) g$ will have at most $|I_1||I_2|$ nonzero terms, which is finite. Thus, RG is closed under multiplication. Furthermore, that RG forms a ring with identity is straightforward. To this end, we make the following definition.

Definition 1.2.1. The ring with identity $(RG, +, \times)$ as outlined above is defined to be the *group ring* of the ring R over the group G.

We will momentarily give an equivalent definition, but we will first need some auxiliary definitions. Again, we take G and R to be a group and a ring with identity, respectively.

Definition 1.2.2. For a function $f\colon G \to R$, we define the *support* of f to be the set

$$\operatorname{supp}(f) = \{x \in \operatorname{Dom}(f) \mid f(x) \neq 0_R\}.$$

Note that the domain of a function can therefore be expressed as the disjoint union of its support and its kernel.

Definition 1.2.3. Let $f_1, f_2\colon G \to R$ be any functions. We define the *convolution* of f_1 and f_2, denoted $f_1 * f_2$, to be the function from G to R given by

$$(f_1 * f_2)(g) := \sum_{h \in G} f_1(h) f_2(h^{-1}g).$$

We aim to give an equivalent definition of the group ring. Consider the set $S := \{f\colon G \to R \mid f \text{ has finite support}\}$. We equip this set with two binary operations, addition and multiplication, defined for all $g \in G$ by

$$(f_1 + f_2)(g) := f_1(g) +_R f_2(g)$$

where $+_R$ is the ring addition from R, and

$$(f_1 \times f_2)(g) := (f_1 * f_2)(g).$$

These operations make S into a ring with identity.

Next, consider any $\alpha = \sum_{g \in G} r_g g$ in RG. Define the function $f_\alpha\colon G \to R$ by $g \mapsto r_g$. Now, the mapping $\phi\colon RG \to S$ whereby $\phi(\alpha) = f_\alpha$ for all $\alpha \in RG$ forms an obvious ring isomorphism. With both RG and S under our belts, we have two equivalent ways of conceptualizing group rings.

Notice that since elements of group rings are finite linear combinations of group elements with ring coefficients, we will often informally write an element of RG as an explicit finite sum $\sum_{i=1}^{n} r_i g_i$, for some $n \in \mathbb{Z}^+$. We will, however, need to be a bit more careful when we add or multiply with this new notation. Indeed, for arbitrary $\sum_{i=1}^{n} r_i g_i$ and $\sum_{i=1}^{m} s_i h_i$, it is not immediately clear how to proceed. However, we ameliorate the situation as follows.

Suppose $\alpha = \sum_{g \in I_1} r_g g = \sum_{i=1}^{n} r_i g_i$ and $\beta = \sum_{h \in I_2} s_h h = \sum_{i=1}^{m} s_i h_i$. Letting $N = \max\{n, m\}$ and letting $I = I_1 \cup I_2$, we see that

$$\alpha + \beta = \sum_{\gamma \in I} (r_\gamma + s_\gamma)\gamma = \sum_{i=1}^{N} (r_i + s_i)\gamma_i$$

where coefficients are appropriately taken to be 0_R where needed and where each $\gamma_i \in \{g_1, \ldots, g_n, h_1, \ldots, h_m\} = I$. Keeping the same N from above, we also have

$$\alpha \times \beta = \left(\sum_{i=1}^{N} r_i \gamma_i\right)\left(\sum_{i=1}^{N} s_i \gamma_i\right) = \sum_{k=1}^{N}\left(\sum_{i,j \text{ such that } \gamma_i\gamma_j = \gamma_k} r_i s_j\right)\gamma_k$$

because if $\gamma_i\gamma_j = \gamma_k$ then $\gamma_i = \gamma_k\gamma_j^{-1}$, matching our original definition of multiplication.

To this effect, since we: can choose the larger of n or m - assume without loss of generality that it is n - and then add a total of $n - m$ zeroes; can take the union of I_1 and I_2 to account for some finite number of group elements; and, note that because multiplication and gathering like terms amounts to the convolution of coefficient maps, we write two arbitrary elements of RG to be $\alpha = \sum_{i=1}^{n} r_i g_i$ and $\beta = \sum_{i=1}^{n} s_i g_i$. We will take this stance throughout the remainder of the thesis.

We conclude this section with a brief aside aimed at those readers who may be familiar with module theory. We mention that an equivalent way of thinking about group rings is that the group ring RG is a free R-module with basis G.

If $r_i = 1_R$, then the element $r_i g_i$ will simply be written as g_i. Similarly, if $g_i = 1_G$, then the element $r_i g_i$ will simply be written r_i. In this fashion, we observe that RG contains isomorphic copies of both R and G.

1.2.1 Examples of Group Rings

Suppose our ring R is the ring of integers $\mathbb{Z}$ and our group G is the free group on two elements $F(a,b)$. Three elements of this group ring are given by

$$\alpha = a - 2b - 7a^{-1}b^{3},$$

$$\beta = -2a^{3}b^{-2} + 13,$$

$$\text{and } \gamma = 2b - 3a^{7}b^{-1}.$$

We then observe that

$$\alpha + \gamma = (a - 2b - 7a^{-1}b^{3}) + (2b - 3a^{7}b^{-1}) = a - 7a^{-1}b^{3} - 3a^{7}b^{-1},$$

that

$$\begin{aligned} \gamma \times \alpha &= (2b - 3a^{7}b^{-1})(a - 2b - 7a^{-1}b^{3}) \\ &= 2ba - 4b^{2} - 14ba^{-1}b^{3} - 3a^{7}b^{-1}a + 6a^{7} + 21a^{7}b^{-1}a^{-1}b^{3}, \end{aligned}$$

and that

$$\beta \times \gamma = (-2a^{3}b^{-2} + 13)(2b - 3a^{7}b^{-1}) = -4a^{3}b^{-1} + 26b + 6a^{3}b^{-2}a^{7}b^{-1} - 39a^{7}b^{-1}.$$

Another example of a group ring is formed from the ring of complex numbers $\mathbb{C}$ and the group of integers $\mathbb{Z}$. In fact, we will take our time with this group ring and show that it is indeed isomorphic to another ring. We start with a definition.

Definition 1.2.4. For an arbitrary field $\mathbb{F}$, the *ring of Laurent polynomials*, which we denote by $\mathbb{F}[t, t^{-1}]$, is the ring with underlying set

$$\mathbb{F}[t, t^{-1}] = \left\{ \sum_{i=-n}^{n} a_i t^i \mid a_i \in \mathbb{F} \text{ and } n \in \mathbb{Z}_{\geq 0} \right\}$$

with addition and multiplication given by ordinary polynomial addition and polynomial multiplication, respectively, but where we allow exponents of the indeterminate to be negative.

Note that $\mathbb{F}[t, t^{-1}]$ indeed forms an $\mathbb{F}$-vector space with basis $\{t^k \mid k \in \mathbb{Z}\}$. In addition, the ring of polynomials $\mathbb{F}[t]$ is a subring of the Laurent polynomials.

We now make the following claim:

Claim. For any field $\mathbb{F}$, the group ring $\mathbb{F}\mathbb{Z}$ is isomorphic to the ring of Laurent polynomials over $\mathbb{F}$, $\mathbb{F}[t, t^{-1}]$.

Proof. Consider the map $f \colon \mathbb{F}\mathbb{Z} \to \mathbb{F}[t, t^{-1}]$ given by

$$\sum_{i=1}^{n} z_i m_i \mapsto \sum_{i=1}^{n} z_i t^{m_i}$$

for $n \in \mathbb{Z}_{>0}$, $z_i \in \mathbb{F}$ and $m_i \in \mathbb{Z}$ for all $1 \leq i \leq n$.

To show that f is well-defined, consider $\alpha = \sum_{i=1}^{n} z_i m_i$ and $\beta = \sum_{i=1}^{n} z_i' m_i'$ in $\mathbb{F}\mathbb{Z}$ for which $\alpha = \beta$. Note that if $\alpha = \beta$, it must be the case that for all $1 \leq i \leq n$ we have $z_i = z_i'$ and $m_i = m_i'$. Now, we observe that

$$
\begin{aligned}
f(\alpha) &= f\left(\sum_{i=1}^{n} z_i m_i\right) \\
&= \sum_{i=1}^{n} z_i t^{m_i} \\
&= \sum_{i=1}^{n} z_i' t^{m_i'} \\
&= f\left(\sum_{i=1}^{n} z_i' m_i'\right) \\
&= f(\beta).
\end{aligned}
$$

This shows that f is well-defined, and we are free now to call it a function.

Now, we aim to show that f is bijective.

Consider any $\alpha, \beta \in \mathbb{FZ}$ - with the same definitions as above - for which $f(\alpha) = f(\beta)$. We see that

$$
\begin{aligned}
f(\alpha) = f(\beta) &\implies f\left(\sum_{i=1}^{n} z_i m_i\right) = f\left(\sum_{i=1}^{n} z_i' m_i'\right) \\
&\implies \sum_{i=1}^{n} z_i t^{m_i} = \sum_{i=1}^{n} z_i' t^{m_i'} \\
&\implies z_i = z_i' \text{ and } m_i = m_i' \text{ for all } 1 \leq i \leq n \\
&\implies \sum_{i=1}^{n} z_i m_i = \sum_{i=1}^{n} z_i' m_i' \\
&\implies \alpha = \beta
\end{aligned}
$$

which shows that f is injective.

Now, consider any $\alpha = \sum_{i=-n}^{n} z_i t^{m_i}$ in $\mathbb{F}[t, t^{-1}]$. Consider the element $\alpha' \in \mathbb{FZ}$

defined by $\alpha' = \sum_{i=0}^{2n} z_{i-n} m_{i-n}$. Indeed, we notice that

$$f(\alpha') = f\left(\sum_{i=0}^{2n} z_{i-n} m_{i-n}\right) = \sum_{i=-n}^{n} z_i t^{m_i} = \alpha.$$

This shows that for all $\alpha \in \mathbb{F}[t, t^{-1}]$ there exists some $\alpha' \in \mathbb{F}\mathbb{Z}$ for which $f(\alpha') = \alpha$, showing that f is surjective. Furthermore, we conclude that, since our function is also injective, f is bijective. All that's left to show is that f forms a ring homomorphism.

Consider $\alpha = \sum_{i=1}^{n} z_i m_i$ and $\beta = \sum_{i=1}^{n} z_i' m_i$ in $\mathbb{F}\mathbb{Z}$. Note

$$\begin{aligned}
f(\alpha + \beta) &= f\left(\sum_{i=1}^{n} z_i m_i + \sum_{i=1}^{n} z_i' m_i\right) \\
&= f\left(\sum_{i=1}^{n} (z_i + z_i') m_i\right) \\
&= \sum_{i=1}^{n} (z_i + z_i') t^{m_i} \\
&= \sum_{i=1}^{n} z_i t^{m_i} + \sum_{i=1}^{n} z_i' t^{m_i} \\
&= f\left(\sum_{i=1}^{n} z_i m_i\right) + f\left(\sum_{i=1}^{n} z_i' m_i\right) \\
&= f(\alpha) + f(\beta).
\end{aligned}$$

Furthermore, we observe that

$$
\begin{aligned}
f(\alpha \times \beta) &= f\left(\left(\sum_{i=1}^{n} z_i m_i\right) \times \left(\sum_{i=1}^{n} z_i' m_i\right)\right) \\
&= f\left(\sum_{k=1}^{n}\left(\sum_{i,j \text{ such that } m_i m_j = m_k} z_i z_j'\right) m_k\right) \\
&= \sum_{k=1}^{n}\left(\sum_{i,j \text{ such that } m_i m_j = m_k} z_i z_j'\right) t^{m_k} \\
&= \left(\sum_{i=1}^{n} z_i t^{m_i}\right)\left(\sum_{i=1}^{n} z_i' t^{m_i}\right) \\
&= f\left(\sum_{i=1}^{n} z_i m_i\right) f\left(\sum_{i=1}^{n} z_i' m_i\right) \\
&= f(\alpha) f(\beta).
\end{aligned}
$$

showing that f is a ring homomorphism. Furthermore, since it is bijective, f is a ring isomorphism. We conclude that $\mathbb{F}\mathbb{Z} \cong \mathbb{F}[t, t^{-1}]$, and the proof is complete. ■

An obvious corollary that we get from this is that the ring of Laurent polynomials over the complex numbers - $\mathbb{C}[t, t^{-1}]$ - is isomorphic to the group ring $\mathbb{C}\mathbb{Z}$.

1.3 Units, Zero-Divisors and Idempotents

We are almost ready to state Kaplansky's Conjectures, but we need a few more key terms.

For the following, we emphasize that R is a commutative ring with identity 1_R.

Definition 1.3.1. A *unit* is a ring element that has a multiplicative inverse - that is, an element $u \in R$ is a unit if there exists some element $u' \in R$ for which $u \times u' = 1_R$ and for which $u' \times u = 1_R$. The inverse element u' will often be denoted u^{-1}.

Definition 1.3.2. A *zero-divisor* is a non-zero ring element r for which there exists another non-zero ring element r' satisfying $r \times r' = 0_R$.

Definition 1.3.3. An *idempotent* is a ring element r that satisfies the equation $r \times r = r$.

At first glance, it might seem like the only elements that satisfy the equation $x^2 = x$ are 1 and 0. While these certainly are solutions to this equation, there may be non-trivial examples. For example, in the ring $M_2(\mathbb{R})$ of 2×2 real matrices, the matrix given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is an idempotent, as is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $M_3(\mathbb{R})$

Of course, two things that both of these rings have in common is that they are not integral domains; if R is an integral domain then indeed only the trivial idempotents of 1 and 0 satisfy $x^2 = x$.

Since these are all ring-theoretical properties, it is natural to ask when a group ring has non-trivial units, zero-divisors and/or idempotents. For example, if we form a group ring RG using a finite group, then we know for certain that RG will have non-trivial zero-divisors. Indeed, pick any group element g with order $n \in \mathbb{Z}$ with $n > 1$. In the group ring, as is the case with every ring, subtraction is well-defined by $a - b = a + (-1) \times b$, where -1 is the additive inverse of the multiplicative identity.

We thus observe that

$$(1_R - g)(1_R + g + g^2 + \cdots + g^{n-1}) = 1_R - 1_R = 0_R = 0_{RG}$$

Indeed, since $|g| = n \neq 1$, we know $g \neq e_G$, hence $1 - g \neq 0$. Similarly, $g^2 \neq e_G$, $g^3 \neq e_G$, $\ldots$, $g^{n-1} \neq e_G$, which tells us that both sums on the left-hand and right-hand sides are non-trivial, so we do indeed have non-trivial zero-divisors in RG.

So, we will always have non-trivial zero-divisors when G has torsion.

Definition 1.3.4. Let G be a group. An element g of G that has finite order is said to be a *torsion element* of G. A group in which every element is a torsion element is said to be a *torsion group*. On the other hand, a group is said to be *torsion-free* if it has no non-trivial torsion elements; in other words, G is torsion-free if every element of G - except for e_G - has infinite order.

Using this new vocabulary, if G has *even one* torsion element, then the group ring RG will have a non-trivial zero-divisor.

So, if RG were to lack zero-divisors, then G would have to be not only infinite, but torsion-free as well. As an aside, we mention that there are indeed infinite groups that are *not* torsion-free. One example is the product of the additive groups $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. In this infinite group, the element $(0, 1 + 3\mathbb{Z})$ is not the identity element and has order 3.

As for units, we quickly realize that if r is any unit in the ring R, then the element $rg \in RG$ is a unit with inverse $r^{-1}g^{-1}$. This is not very interesting though, and so we call these types of group ring elements *trivial units*.

1.4 Kaplansky's Conjectures

In this context, Irving Kaplansky proposed a number of interlinked conjectures concerning the ring-theoretical properties of the group ring. Specifically:

Claim. Assume that G is a torsion-free group and that R is an integral domain.

1. *Unit Conjecture.* The group ring RG contains no non-trivial units.

2. *Zero-Divisor Conjecture.* The group ring RG contains no non-trivial zero divisors; equivalently, the group ring RG is itself an integral domain.

3. *Idempotent.* The group ring RG contains no non-trivial idempotents.

At first, these conjectures may seem like dissimilar statements, but they are actually quite related. Specifically, the Unit Conjecture implies the Zero-Divisor Conjecture, which in turn implies the Idempotent Conjecture. Informally,

$$\text{Unit Conjecture} \implies \text{Zero-Divisor Conjecture} \implies \text{Idempotent Conjecture.}$$

We will prove this chain of implications shortly, but we will dedicate a brief word to the semantics of the statements themselves. The statements themselves refer to group rings in general, but individual counterexamples may be brought forward that disprove a specific conjecture. In a noteworthy example - [Gar21] - Giles Gardam constructed a counterexample to the Unit Conjecture; Gardam used a field of characteristic 2 as his ring and for his group used a type of crystallographic group. He constructed two non-trivial elements that did indeed multiply to 1. Despite this counterexample, the Zero-Divisor and Idempotent Conjectures remained unsolved. In particular, the Kaplansky conjectures are open for group rings RG where R has characteristic 0.

To this effect, we prove the following implication, which is the simpler of the two. The proof is common knowledge by now, but we include a detailed proof for completion.

Theorem 1.4.1. *Suppose that* Γ *is a torsion-free group and that* R *is an integral domain. If the group ring* $R\Gamma$ *has no non-trivial zero-divisors, then* $R\Gamma$ *has no non-trivial idempotents.*

Proof. Suppose that Γ is a torsion-free group and that R is an integral domain. We proceed by contraposition. Assume that the group ring $R\Gamma$ *does* have a non-trivial idempotent; that is, there exists some $x \in R\Gamma$, with $x \neq 1_{R\Gamma}$ and $x \neq 0_{R\Gamma}$ that satisfies $x^2 = x$. Notice then that $x^2 - x = 0_{R\Gamma}$, and furthermore that $x(x-1) = 0_{R\Gamma}$. Now, since x is neither $0_{R\Gamma}$ nor $1_{R\Gamma}$, we have shown two non-trivial zero-divisors in both x and $x - 1$, and so $R\Gamma$ has non-trivial zero-divisors. ■

The proof that the Unit Conjecture implies the Zero-Divisor Conjecture on the other hand needs to be built up to, and we will need some background properties to do so.

The following definitions are standard and we give them for clarity and completion's sake.

Definition 1.4.1. Let R be a ring. We say that R is *prime* if, given $a, b \in R$ such that $arb = 0_R$ for all $r \in R$, then either $a = 0_R$ or $b = 0_R$.

Definition 1.4.2. Let G be an infinite group. We will say that G is *prime* if it contains no finite normal subgroup other than the trivial subgroup.

The arguments we make here hinge on Connell's work regarding certain algebraic properties of the group ring that are based on properties it inherits from its constituent group and ring.

Theorem 1.4.2. *If* Γ *is a torsion-free group, then* Γ *is prime.*

Proof. If Γ is torsion-free, then it has no finite subgroups, save for the trivial subgroup. Indeed, if it did have a finite subgroup, then an element in that subgroup would have finite order - we would have a torsion element. This, by definition, makes Γ a prime group in the sense of Connell, and we are done. ■

Theorem 1.4.3. *Suppose R is a ring with identity $1_R \neq 0_R$. Then R is an integral domain if and only if it is a commutative prime ring.*

Proof. ($\Longleftarrow$) Suppose that R is a commutative prime ring. Suppose that a and b in R and that $arb = 0_R$ for all $r \in R$ implies $a = 0_R$ or $b = 0_R$. Taking $r = 1_R$, we see that $a \cdot 1_R \cdot b = ab = 0_R$ implies that $a = 0_R$ or $b = 0_R$. By definition, this property coupled with the added assumption of commutativity makes R an integral domain.

($\Longrightarrow$) Suppose that R is an integral domain. This means that R is a commutative ring where $ab = 0_R$ implies that either $a = 0_R$ or $b = 0_R$. Now, take any $r \in R$. Note that if we have the equation $ab = 0_R$, then multiplying both sides by r gets us $abr = 0_R \cdot r = 0_R$. Since R is assumed to be an integral domain, it is commutative. This means that $abr = arb$. So, $arb = 0_R$ implies that $a = 0_R$ or $b = 0_R$. Since $r \in R$ was taken to be arbitrary, we furthermore have that $arb = 0_R$ for *all* $r \in R$ implies that $a = 0_R$ or $b = 0_R$. By definition, this makes R a prime ring that is also commutative. ■

Connell showed that a group ring is a prime ring if and only if its constituent group is prime as is its constituent ring; we state this theorem without proof.

Theorem 1.4.4. *[Con63] Suppose R is a commutative ring with identity and G is a group. The group ring RG is prime if and only if R is prime and G is prime.*

Corollary. Suppose R is an integral domain and Γ is a torsion-free group. Then the group ring $R\Gamma$ is prime.

Proof. Per Connell's results, it suffices to show that both R and Γ are prime. We showed in Theorems 1.4.2 and 1.4.3, respectively, that a torsion-free group is prime and that an integral domain is a prime ring. We easily conclude that the group ring $R\Gamma$ is prime itself. ■

We had to build up to such a short proof, but the previously stated corollary will be invaluable in showing that the Unit Conjecture implies the Zero-Divisor Conjecture. To this end, we prove the following theorem, which is standard folklore.

Theorem 1.4.5. *Suppose R is an integral domain and Γ is a torsion-free group. If the group ring $R\Gamma$ has no non-trivial units, then $R\Gamma$ has no non-trivial zero-divisors.*

Proof. We proceed by contraposition. Suppose that $R\Gamma$ has non-trivial zero-divisors; that is, suppose $x, y \in R\Gamma$, $x \neq 0_{R\Gamma}$, $y \neq 0_{R\Gamma}$, and $xy = 0_{R\Gamma}$. Now, we know that $R\Gamma$ is prime by the above corollary. So, we know there exists $z \in R\Gamma$ for which $yzx \neq 0_{R\Gamma}$. Now, observe that

$$(yzx)^2 = (yzx)(yzx) = (yz)(xy)(zx) = (yz) \cdot 0_{R\Gamma} \cdot (zx) = 0_{R\Gamma}.$$

Therefore, we have that

$$(1 + yzx)(1 - yzx) = 1 - (yzx)^2 = 1 - 0 = 1.$$

We know yzx is non-zero, so all that is left is to show that $1 \pm yzx$ are non-trivial. Suppose not; for instance, if $1 \pm yzx = r\gamma$ for some $r \in R^\times$ a unit and non-trivial $\gamma \in \Gamma$. Then

$$0 = (yzx)^2 = (1 \pm r\gamma)^2 = 1 \pm 2r\gamma + r^2\gamma^2$$

which is impossible because 1 and r are non-zero in R and because e_Γ, γ and γ^2 are distinct in Γ due to the group's lack of torsion. Indeed, we see that $1 + yzx$ and

$1 - yzx$ are non-trivial units, and the proof is complete. ■

Now that we know the relationships between the Kaplansky Conjectures, the rest of the thesis will be dedicated to showing when a certain group ring satisfies these conjectures, based on which groups and rings constitute the group ring.

Chapter 2

Unique Products in Groups

In this chapter we introduce a new structure we can impose on groups that will furthermore satisfy the Kaplansky conjectures.

2.1 One Unique Product or Two?

Before we delve into definitions, we aim to cite Bowditch as the source of this exposition, and can be found in [Bow00].

Definition 2.1.1. Suppose G is a group, and that $A \subseteq G$ and $B \subseteq G$. We define the set AB by $AB := \{ab \mid a \in A \text{ and } b \in B\}$.

This simple definition will be used in the following definition that will be of utmost importance.

Definition 2.1.2. A group G is said to satisfy the *unique product property* whenever, given any two finite non-empty subsets A and B of G, there exists $x \in AB$ such that $x = ab$ for unique $(a, b) \in A \times B$.

Despite how straightforward of an idea the unique product property is, it is actually enough to satisfy one of the Kaplansky Conjectures. For instance, the following theorem is attributed to Passman.

Theorem 2.1.1. *[Pas71] Suppose that* Γ *is a torsion-free group and that* R *is any integral domain. If* Γ *satisfies the unique product property, then the group ring* $R\Gamma$ *has no non-trivial zero divisors.*

Proof. Suppose that R is an integral domain and that Γ is a torsion-free group. Assume further that Γ has the unique product property.

By way of contradiction, suppose there exists non-trivial zero divisors in the group ring $R\Gamma$; say $\alpha \in R\Gamma$ and $\beta \in R\Gamma$. Suppose further that $\alpha = \sum_{i=1}^{m} r_i\gamma_i$ and that $\beta = \sum_{j=1}^{n} s_j\delta_j$. We will assume that each of the r_i and each of the s_j are all nonzero. In this manner, we will let $A = \{\gamma_i\}_{i=1}^{m}$ and we will let $B = \{\delta_j\}_{j=1}^{n}$. Note that $|A| \geq 2$ and $|B| \geq 2$, since α and β are non-trivial zero divisors, and hence must be at least binomials in the group ring.

Now, we observe that

$$\begin{aligned} 0 &= \alpha\beta \\ &= \left(\sum_{i=1}^{m} r_i\gamma_i\right)\left(\sum_{j=1}^{n} s_j\delta_j\right) \\ &= \sum_{\gamma\in\Gamma}\left(\sum_{\gamma_i\in A, \delta_j\in B \text{ such that } \gamma_i\delta_j=\gamma} r_i s_j\right)\gamma \end{aligned}$$

where an empty sum is 0_R by convention.

Now, as $|A| \geq 2$ and $|B| \geq 2$, we may invoke the unique product property of Γ. We know there to exist some $\varepsilon \in AB$ that is uniquely expressable as a product - say

$\varepsilon = \gamma_k \delta_L$, for some $\gamma_k \in A$, some $\delta_L \in B$, and with $1 \leq k \leq m$ and with $1 \leq L \leq n$.

Because of this unique representation as a product, we must have that the coefficient of ε in AB is merely $r_k s_L$. Now, recall that the group ring $R\Gamma$ is a free R-module with basis Γ. Thus, since we have a collection of basis elements that sum to zero, linear independence tells us that all of the scalars from the ring are zero. In particular, we must have the coefficient of ε is 0_R. This tells us that $0_R = r_k s_L$. Since R is taken to be an integral domain, we must therefore have that $r_k = 0_R$ or that $s_L = 0_R$. Either of these cases result in contradiction, since we assumed at the start that all of the r_i and each of the s_j were all nonzero. This means that we cannot possibly have non-trivial zero divisors in $R\Gamma$, thus completing the proof. ■

We now introduce a similar construction, and how it relates to the Kaplansky Conjectures.

Definition 2.1.3. A group G is said to satisfy the *two-unique product property* whenever, given any two finite non-empty subsets A and B of G satisfying $|A| \geq 2$ and $|B| \geq 2$, then there exist two distinct $x_1, x_2 \in AB$ such that $x_1 = a_1 b_1$ for unique $(a_1, b_1) \in A \times B$ and $x_2 = a_2 b_2$ for unique $(a_2, b_2) \in A \times B$.

We similarly see that the Two Unique Product Property implies one of the Kaplansky Conjectures, again attributing the following to Passman.

Theorem 2.1.2. *[Pas71] Suppose that Γ is a torsion-free group and that R is any integral domain. If Γ satisfies the two-unique product property, then the group ring $R\Gamma$ has no non-trivial units.*

Proof. Suppose that R is an integral domain and that Γ is a torsion-free group. Assume further that Γ has the two-unique product property. By way of contradiction, suppose there exist non-trivial units in the group ring $R\Gamma$. Let α and β be two such units with $\alpha\beta = 1$. Now, $\alpha = \sum_{\gamma \in A} r_\gamma \gamma$ and $\beta = \sum_{\delta \in B} s_\delta \delta$, where we assume without loss of generality that all r_γ and all s_δ are nonzero. Furthermore, as α and β are taken to be non-trivial, $|A| \geq 2$ and $|B| \geq 2$.

Now, we observe that

$$\begin{aligned} 1_{R\Gamma} &= \alpha\beta \\ &= \left(\sum_{\gamma\in A} r_\gamma \gamma\right)\left(\sum_{\delta\in B} s_\delta \delta\right) \\ &= \sum_{\varepsilon\in AB}\left(\sum_{\gamma\in A,\delta\in B \text{ such that } \gamma\delta=\varepsilon} r_\gamma s_\delta\right)\varepsilon \\ &= \sum_{\varepsilon\in AB} t_\varepsilon \varepsilon \end{aligned}$$

where the internal sum t_ε is adopted for ease of notation.

Recall that $1_{R\Gamma} := 1_R 1_\Gamma$, and hence in $\sum_{\varepsilon\in AB} t_\varepsilon\varepsilon$ each $t_\varepsilon = 0_R$, except for t_{1_Γ}, in which case $t_{1_\Gamma} = 1_R$.

Now, A and B each have at least two elements, so we may invoke the two-unique product property of Γ. In AB, there thus exist distinct ε_1 and ε_2 that are each uniquely expressable as products - say $\varepsilon_1 = \alpha_1\beta_1$ and that $\varepsilon_2 = \alpha_2\beta_2$, for appropriate such elements in A and B. Now,

$$t_{\varepsilon_1} := \sum_{\gamma\in A,\delta\in B \text{ such that } \gamma\delta=\varepsilon_1} r_\gamma s_\delta = r_{\alpha_1} s_{\beta_1}.$$

Similar reasoning shows that

$$t_{\varepsilon_2} = r_{\alpha_2} s_{\beta_2}.$$

Now, we assumed that all of the r_γ and that all of the s_δ are nonzero ring elements. Since R is an integral domain, we have that $r_{\alpha_1} \neq 0_R$ and $s_{\beta_1} \neq 0_R$ which imply that

$t_{\varepsilon_1} \neq 0_R$, and similarly $r_{\alpha_2} \neq 0_R$ and $s_{\beta_2} \neq 0_R$, implying that $t_{\varepsilon_2} \neq 0_R$.

As mentioned above, each $t_\varepsilon = 0_R$ except for t_{1_Γ}, in which case $t_{1_\Gamma} = 1_R$. This means that $t_{\varepsilon_1} \neq 0_R$ and $t_{\varepsilon_2} \neq 0_R$ can only happen if $t_{\varepsilon_1} = t_{\varepsilon_2} = 1_R = t_{1_\Gamma}$, which implies that $\varepsilon_1 = \varepsilon_2 = 1_\Gamma$, contradicting the assumption that ε_1 and ε_2 are distinct.

It must therefore be the case that there are no non-trivial units in $R\Gamma$, completing the proof. ■

We have thus seen that if a group has the Two Unique Product Property then for a group ring built from this group and any integral domain the Unit Conjecture (and hence the Zero-Divisor and Idempotent Conjectures) will hold. Similarly, if a group has the Unique Product Property then for a group ring built from this group and any integral domain the Zero-Divisor Conjecture (and hence the Idempotent Conjecture) will hold.

As it turns out, the Unique Product Property is equivalent to the Two Unique Property. Indeed, the following theorem is attributed to Strojnowski.

Theorem 2.1.3. *[Str80] Suppose that G is any group. Then G has the Two Unique Product Property if and only if it has the Unique Product Property.*

Proof. The forward implication is easy to see, since if a set has at least two unique products then it has at least one unique product. So, we concern ourselves with the reverse implication; for the ease of the reader, we mention that we proceed on a case-by-case basis.

Assume towards a contradiction that G has the unique product property yet not the two-unique product property. Thus, there exist finite nonempty subsets A and B of G such that $2 < |A| + |B|$ and in the set AB there exists exactly one element ab with a unique representation. Let $C = a^{-1}A = \{a^{-1}\alpha \mid \alpha \in A\}$ and let $D = Bb^{-1} = \{\beta b^{-1} \mid \beta \in B\}$. Now, by definition of these sets, only $1 = 1 \cdot 1 = (a^{-1}a) \cdot (bb^{-1})$ has a unique representation in CD. Next, define the sets $E = D^{-1}C$ and $F = DC^{-1}$, and observe each element from EF can be written in the form $(d_1^{-1}c_1)(d_2c_2^{-1})$.

We argue that $C^{-1} \cap D = \{1\}$. Indeed, consider any $x \in C^{-1} \cap D$. Then $x \in D$ and $x^{-1} \in C$, and so $CD \ni x^{-1}x = 1$. Since only $1 = 1 \cdot 1$ is uniquely representable as a product in CD, we must have that $x = x^{-1} = 1 = 1^{-1}$, hence we see that $C^{-1} \cap D = \{1\}$.

Next, we show that if $c_1 \neq 1$ or $d_2 \neq 1$, then there must exist some $c_3 \in C$ and $d_3 \in D$ such that $c_1d_2 = c_3d_3$ and with $c_1 \neq c_3$. The explanation is pretty straightforward; indeed, if $c_1 \neq 1$ or $d_2 \neq 1$, then we must have $c_1d_2 \neq 1$, following the logic from the preceding paragraph. (Could $c_1^{-1} = d_2$? NO, since the previous paragraph would mean that $c_1^{-1} = d_2$ if and only if $c_1 = d_2 = 1$, which we assume is not the case). Since $c_1d_2 \neq 1$, there must be another way of expressing this value as a product. So, we must have some $c_3 \in C$ and $d_3 \in D$ with $c_1d_2 = c_3d_3$. The fact that $c_1 \neq c_3$ is easy to see, since if $c_1 = c_3$, then we would therefore observe that $d_2 = d_3$, and the equation becomes trivial.

Thus, the element $(d_1^{-1}c_1)(d_2c_2^{-1}) \in EF$ has another representation $(d_1^{-1}c_3)(d_3c_2^{-1})$.

Similarly, if $c_2 \neq 1$ or if $d_1 \neq 1$ then there exists some $c_4 \in C$ and some $d_4 \in D$ such that $c_2d_1 = c_4d_4$ and $c_2 \neq c_4$. The explanation behind this is identical to what we mentioned above.

We see that $d_1^{-1}c_2^{-1} = d_4^{-1}c_4^{-1}$ and hence the element $(d_1^{-1} \cdot 1) \cdot (1 \cdot c_2^{-1})$ has another representation in $(d_4^{-1} \cdot 1) \cdot (1 \cdot c_4^{-1})$. Now, since $2 < |C| + |D|$ by definition of C and D, there exists some element in $C \cup D$ which is not equal to 1. Assume without loss of generality that $1 \neq c \in C$. Then the element $(1 \cdot 1)(1 \cdot 1)$ from EF has another representation $(1 \cdot c)(1 \cdot c^{-1})$. We see then there is no element in EF which has a unique representation in the form xy where $x \in E$ and $y \in F$. This is a contradiction of the earlier assumption that G is a unique product group, and therefore we must have that G is also a two-unique product group. ■

This now allows us to use the Unique and Two Unique Product Properties interchangeably. For example, we now know that if G is a torsion-free group that has the Two Unique Product Property, then for any integral domain R the group ring RG satisfies the Zero-Divisor Conjecture, and we can claim this using two streams of implications. Firstly, we know that the Two Unique Product Property implies the Unique Product Property which implies the Zero-Divisor Conjecture; secondly, we know that the Two Unique Product Property implies the Unit Conjecture which implies the Zero-Divisor Conjecture.

Chapter 3

Diffuseness

In this chapter, we begin to delve into material that is more geometric and metric in nature. We show that the Kaplansky Conjectures are upheld when a group ring RG is formed with an integral domain R and a group G that has these properties.

We also mention that this chapter is not only motivated by definitions and ideas put forth by Bowditch, but also consists of an exposition of Bowditch's work.

3.1 Laying the Groundwork: The Set $\Delta(A)$

We will be discussing different types of diffuseness, and the following definition will be paramount to the definitions of the types of diffuseness. These definitions and theorems can be found in [Bow00].

Definition 3.1.1. Let G be any group with identity element e, and suppose $X \subseteq G$ is any subset. We say that the set X is *antisymmetric* if $X \cap X^{-1} \subseteq \{e\}$.

Lemma 3.1.1. *Suppose Γ is a torsion-free group. Given a finite subset A of Γ and arbitrary yet fixed $a \in A$, we prove that Aa^{-1} is an antisymmetric set if and only if for all $1 \neq \gamma \in \Gamma$, we have $\gamma a \notin A$ or $\gamma^{-1} a \notin A$.*

Proof. ($\Longrightarrow$) We proceed by contraposition. Suppose there exists some $\gamma_a \in \Gamma \setminus \{1\}$ such that both $\gamma_a a, \gamma_a^{-1} a \in A$. Notice then that $\gamma_a = \gamma_a a a^{-1} \in A a^{-1}$ since $\gamma_a a \in A$. Similarly, we observe that $\gamma_a = a a^{-1} \gamma_a = a(\gamma_a^{-1} a)^{-1} \in a A^{-1}$ since $\gamma_a^{-1} a \in A$. Combining both of these observations leads us to deduce that $\gamma_a \in A a^{-1} \cap a A^{-1}$. Because $\gamma_a \neq 1$, we see that $A a^{-1} \cap (A a^{-1})^{-1} = A a^{-1} \cap a A^{-1} \not\subseteq \{1\}$, thus showing that $A a^{-1}$ is not antisymmetric.

($\Longleftarrow$) We again proceed by contraposition. Suppose that $A a^{-1}$ is not antisymmetric. By definition, this means that $A a^{-1} \cap (A a^{-1})^{-1} \not\subseteq \{1\}$, or in other words that $A a^{-1} \cap a A^{-1} \not\subseteq \{1\}$. So, we can say for certain that there exists some $x \in A a^{-1} \cap a A^{-1}$ such that $x \neq 1$. Because $x \in A a^{-1}$, we know there to exist some $\alpha_1 \in A$ for which $x = \alpha_1 a^{-1}$. Similarly, we know there to exist some $\alpha_2 \in A$ for which $x = a \alpha_2^{-1}$. Now, we observe that $xa = \alpha_1 a^{-1} a = \alpha_1 \in A$ and also that $x^{-1} a = (a \alpha_2^{-1})^{-1} a = \alpha_2 a^{-1} a = \alpha_2 \in A$. Because $x \neq 1$, we can take $\gamma = x$ and we arrive at the negation of statement 2, completing the proof by contraposition. ∎

Definition 3.1.2. Suppose Γ is a torsion-free group, with $A \subset \Gamma$ a finite subset. The collection of all elements a of A that satisfy either of the two equivalent conditions enumerated in Lemma 3.1.1 will be referred to as $\Delta(A)$, which we will refer to as the *boundary of A*

We remark that the contrapositive of Condition 2 above tells us that $a \notin \Delta(A)$ if and only if there exists some $1_\Gamma \neq \gamma \in \Gamma$ such that both $\gamma a \in A$ and $\gamma^{-1} a \in A$. Thus, we may imagine the set $\Delta(A)$ as the "boundary" of the set A - hence the name - whose elements get "pushed out" of A upon the action of any non-trivial $\gamma \in \Gamma$ or its inverse.

3.2 Diffuseness and Weak Diffuseness

To this effect, we make the following definitions; both are taken from Bowditch's paper [Bow00].

Definition 3.2.1. [Bow00] A group Γ is said to be *diffuse* whenever, for every finite subset $A \subset \Gamma$ satisfying $|A| \geq 2$, we have $|\Delta(A)| \geq 2$.

A similar notion of weak diffuseness also exists.

Definition 3.2.2. [Bow00] A group Γ is said to be *weakly diffuse* whenever, for every nonempty finite subset $A \subset \Gamma$, the set $\Delta(A)$ is nonempty.

At first glance these two properties seem non-comparable; after all, both deal with the size of the set $\Delta(A)$, something which can't exactly change. However, we will soon see that these definitions are actually equivalent.

Before we prove this claim, we will provide some auxiliary lemmas and their proofs.

Lemma 3.2.1. *Suppose X and Y are sets and that $A, B \subseteq X$ are any subsets. Suppose also that $f \colon X \to Y$ is an injective function. We prove that $f(A \cap B) = f(A) \cap f(B)$.*

Proof. First, consider any $y \in f(A) \cap f(B)$. By definition of intersection, we see that $y \in f(A)$ and that $y \in f(B)$. By definition of image, there exists some $a_y \in A$ and some $b_y \in B$ for which $f(a_y) = y$ and for which $f(b_y) = y$. Since f is injective by assumption, this means that $a_y = b_y$. As we assumed that $a_y \in A$ and that $b_y \in B$, we observe that $a_y = b_y \in A \cap B$, and by definition of image, we see that $f(A \cap B) \ni f(b_y) = f(a_y) = y$, showing that $f(A \cap B) \supseteq f(A) \cap f(B)$.

Next, consider any $y \in f(A \cap B)$. By definition of image, there exists some $x_y \in A \cap B$ for which $f(x_y) = y$. Now, because $x_y \in A \cap B$, we see that $x_y \in A$ and that $x_y \in B$, by definition of intersection. By definition of image, we have that $y = f(x_y) \in f(A)$

and also that $y = f(x_y) \in f(B)$, thus showing that $y \in f(A) \cap f(B)$. This shows that $f(A \cap B) \subseteq f(A) \cap f(B)$, completing the proof. ∎

Lemma 3.2.2. *If $A, B \subset \Gamma$ are any nonempty, finite subsets of the group Γ, then $\Delta(A \cup B) \subseteq \Delta(A) \cup \Delta(B)$.*

Proof. Consider any $x \in \Delta(A \cup B) \subseteq A \cup B$. We will consider two cases.

First, we assume $x \in A$. Now, we consider two subcases. Suppose $x \in \Delta(A)$. If this is true, then $x \in \Delta(A) \cup \Delta(B)$, and we are done. Next, assume that $x \notin \Delta(A)$. By definition of this set, this means that there exists some $\gamma_a \in \Gamma$ with $1 \neq \gamma_a$ such that $\gamma_a x \in A$ and that $\gamma_a^{-1} x \in A$. Since $x \in \Delta(A \cup B)$, this means that for every $\gamma \neq 1$ in Γ, we see that $\gamma x \notin A \cup B$ or that $\gamma^{-1} x \notin A \cup B$. But $\gamma_a x, \gamma_a^{-1} x \in A \subset A \cup B$, thus implying $x \notin \Delta(A \cup B)$, contradicting assumption. So, if $x \in A$, then it *must* also be in $\Delta(A)$, for if not then a contradiction is reached.

For the second case, we assume $x \in B$. This argument is symmetric to that above.

As x was taken to be arbitrary, we see that $\Delta(A \cup B) \subseteq \Delta(A) \cup \Delta(B)$, and the proof is complete. ∎

Lemma 3.2.3. *Suppose $A \subset \Gamma$ and that $g \in \Gamma$ is arbitrary yet fixed. We prove that $\Delta(Ag) = \Delta(A)g$.*

Proof. First, consider any $x \in \Delta(Ag) \subseteq Ag = \{ag \mid a \in A\}$. This means that for all $\gamma \in \Gamma \setminus \{1\}$, we observe that $\gamma x \notin Ag$ or that $\gamma^{-1} x \notin Ag$. We saw already that $x \in Ag$, so we can say that there exists some $a_x \in A$ for which $x = a_x g$. Now, for all $\gamma \in \Gamma \setminus \{1\}$, we have that $\gamma a_x g \notin Ag$. This is only possible if $\gamma a_x \notin A$ for all $\gamma \in \Gamma \setminus \{1\}$. By definition, this makes $a_x \in \Delta(A)$. Multiplying on the right by g yields $a_x g \in \Delta(A)g$, hence $x \in \Delta(A)g$. Since x was taken to be arbitrary, we have $\Delta(Ag) \subseteq \Delta(A)g$.

Now, consider any $x \in \Delta(A)g$. This means that there exists some $a_x \in \Delta(A)$ such that $x = a_x g$. Since $a_x \in \Delta(A)$, $\gamma a_x \notin A$ or $\gamma^{-1} a_x \notin A$ for all $\gamma \in \Gamma \setminus \{1\}$. Multiplying on the right by g yields $\gamma a_x g \notin Ag$ or $\gamma^{-1} a_x g \notin Ag$ for all $\gamma \in \Gamma \setminus \{1\}$. This means, by definition, that $x \in \Delta(Ag)$. Since x was taken to be arbitrary, we see that $\Delta(A)g \subseteq \Delta(Ag)$.

This shows that $\Delta(Ag) \subseteq \Delta(A)g \subseteq \Delta(Ag)$, thus making $\Delta(Ag) = \Delta(A)g$, and the proof is complete. ∎

Lemma 3.2.4. *For a group Γ, suppose $A \subseteq \Gamma$ is any subset and $a \in \Gamma$ is any element. We prove that $a^{-1}A \cap A^{-1}a = a^{-1}(aA^{-1} \cap Aa^{-1})a$.*

Proof. Recall that conjugation forms a group automorphism. That is to say, for any group G and for all $g \in G$, the map $f_g \colon G \to G$ defined by $x \mapsto g^{-1}xg$ forms a group isomorphism. Now, a group isomorphism is injective, and so we invoke the Lemma 3.2.1 to observe the following.

$$\begin{aligned} a^{-1}(aA^{-1} \cap Aa^{-1})a &= f_a(aA^{-1} \cap Aa^{-1}) \\ &= f_a(aA^{-1}) \cap f_a(Aa^{-1}) \\ &= a^{-1}(aA^{-1})a \cap a^{-1}(Aa^{-1})a \\ &= A^{-1}a \cap a^{-1}A \\ &= a^{-1}A \cap A^{-1}a \end{aligned}$$

∎

Lemma 3.2.5. *For a group Γ, suppose $A \subset \Gamma$ is any finite subset with $|A| \geq 2$. We prove that $a \in \Delta(A)$ if and only if $a^{-1} \in \Delta(A^{-1})$.*

Proof. ($\Longrightarrow$) Suppose $a \in \Delta(A)$. By definition, this makes the set Aa^{-1} antisymmetric. By definition of antisymmetric, $Aa^{-1} \cap (Aa^{-1})^{-1} \subseteq \{1\}$, or in other words that $Aa^{-1} \cap aA^{-1} \subseteq \{1\}$. To show that $a^{-1} \in \Delta(A^{-1})$, we need to have the set $A^{-1}(a^{-1})^{-1} = A^{-1}a$ antisymmetric. This means $A^{-1}a \cap (A^{-1}a)^{-1} \subseteq \{1\}$, or in other words that $A^{-1}a \cap a^{-1}A \subseteq \{1\}$. By Lemma 3.2.4, we know that $a^{-1}A \cap A^{-1}a = a^{-1}(aA^{-1} \cap Aa^{-1})a$, so we can equivalently show that $a^{-1}(aA^{-1} \cap Aa^{-1})a \subseteq \{1\}$.

By assumption, $aA^{-1} \cap Aa^{-1} \subseteq \{1\}$. We consider two cases.

First, suppose $aA^{-1} \cap Aa^{-1} = \emptyset$. Then

$$a^{-1}(aA^{-1} \cap Aa^{-1})a = a^{-1}(\emptyset)a = \emptyset \subseteq \{1\}.$$

Second, suppose $aA^{-1} \cap Aa^{-1} = \{1\}$. Then

$$\begin{aligned} a^{-1}(aA^{-1} \cap Aa^{-1})a &= a^{-1}(\{1\})a \\ &= \{a^{-1} \cdot 1 \cdot a\} \\ &= \{1\} \\ &\subseteq \{1\}. \end{aligned}$$

In both cases, we see that $a^{-1}(aA^{-1} \cap Aa^{-1})a \subseteq \{1\}$. Invoking Lemma 3.2.4, this means $a^{-1}A \cap A^{-1}a \subseteq \{1\}$. Hence $A^{-1}(a^{-1})^{-1} \cap (A^{-1}(a^{-1})^{-1})^{-1} \subseteq \{1\}$. This makes $A^{-1}(a^{-1})^{-1}$ an antisymmetric set, and we conclude that $a^{-1} \in \Delta(A^{-1})$ by definition.

($\Longleftarrow$) The proof of this implication is symmetric to the proof of the other implication. ∎

We now set out to prove the following claim, which we state as a theorem.

Theorem 3.2.6. *[LWM14] Suppose* Γ *is torsion-free group. We prove that* Γ *is diffuse if and only if it is weakly diffuse.*

Proof. ($\Longrightarrow$) Suppose Γ is diffuse. This means that $|\Delta(A)| \geq 2$ for all appropriate A. But if $|\Delta(A)| \geq 2$ we must also have $\Delta(A) \neq \emptyset$, which thus makes Γ weakly diffuse.

($\Longleftarrow$) By way of contraposition, suppose that Γ is weakly diffuse yet not diffuse. This means that there exists some finite subset $A \subset \Gamma$ with $|A| \geq 2$ such that $|\Delta(A)| = 1$. By Lemma 3.2.3, we know that for all $g \in \Gamma$, $\Delta(Ag) = \Delta(A)g$. So, if $\Delta(A)$ is just a singleton, then we may as well assume that it is precisely $\{1\}$; for if it is not, say it is $\{\hat{a}\}$. We then observe that

$$\{\hat{a}\} = \{1\}\hat{a} = \Delta(A)\hat{a} = \Delta(A\hat{a}).$$

In this manner, any singleton can be "scaled" to achieve it from the singleton $\{1\}$.

Now, by Lemma 3.2.5, we know that $a \in \Delta(A)$ if and only if $a^{-1} \in \Delta(A^{-1})$. So, if we assume that $\Delta(A) = \{1\}$ it must also be the case that $\Delta(A^{-1}) = \{1^{-1}\} = \{1\}$. By Lemma 3.2.2, we know that

$$\Delta(A \cup A^{-1}) \subseteq \Delta(A) \cup \Delta(A^{-1}) = \{1\} \cup \{1\} = \{1\}.$$

Suppose then that $1 \in \Delta(A \cup A^{-1})$. However, if $\gamma \in A$ is any element with $\gamma \neq 1$ (which we know there to be at least one of since the set A has at least two elements), we then observe that

$$\gamma^{\pm 1} \cdot 1 = \gamma^{\pm 1} \in A \cup A^{-1}.$$

This contradicts the assumption that $1 \in \Delta(A \cup A^{-1})$. Because we know that $\Delta(A \cup A^{-1}) \subseteq \{1\}$, the only possibility that remains is that $\Delta(A \cup A^{-1}) = \emptyset$.

However, this contradicts the hypothesis that Γ is weakly diffuse. Therefore, we *must* have that Γ is diffuse, completing the proof. ■

With this knowledge at our disposal, we may interchangeably use diffuse and weakly diffuse from here on out.

As is the case with so many claims we have encountered thus far, we see that diffuseness is related to the Kaplansky Conjectures. The following theorem is attributed to Bowditch.

Theorem 3.2.7. *[Bow00] Diffuse groups satisfy the two-unique-product property.*

Proof. Consider any diffuse group Γ, with finite subsets $A, B \subseteq \Gamma$ for which $2 \leq |A|$ and $2 \leq |B|$. Let $C = AB = \{ab \mid a \in A \text{ and } b \in B\}$. Note that by definition of A and B, we know that $|C| \geq 2$ - to see this, we can say for certain that there are at least two elements of each set A and B, call them $a, a' \in A$ and $b, b' \in B$ with $a \neq a'$ and $b \neq b'$. It then follows that $ab \neq a'b$, and these are two distinct elements in C. Now, since we assume Γ to be diffuse, we see that $|\Delta(C)| \geq 2$ too.

We aim to show that if $c \in \Delta(C)$, then c has a unique expression as a product. By way of contradiction, suppose that $c = a_0b_0 = a_1b_1$, where $a_0, a_1 \in A$, $b_0, b_1 \in B$, and $a_0 \neq a_1$. Let $\gamma = a_0a_1^{-1} \in \Gamma$, and notice that $\gamma \neq 1_\Gamma$ since we assume $a_0 \neq a_1$.

Note that $\gamma c = a_0a_1^{-1}c = a_0a_1^{-1}(a_1b_1) = a_0b_1 \in C$, and also that $\gamma^{-1}c = a_1a_0^{-1}c = a_1a_0^{-1}(a_0b_0) = a_1b_0 \in C$. This contradicts the assumption that $c \in \Delta(C)$. This means that Γ satisfies the two-unique-product property. ■

3.3 Group Actions

It is at this point where our material makes a shift. Everything we have done thus far has been quite algebraic in nature, and while that remains true, we start to focus heavily on groups acting on different sets and, specifically, metric spaces. Because we rely very much on group actions throughout the rest of the thesis, we use this section to refresh our memory about group actions, as well as give a few definitions of specific types of actions we plan to use.

There are a few different ways of thinking about group actions. The most well known should be familiar to undergraduate students who have taken an introduction to group theory course; for instance, Dummit and Foote is a good reference. [DF04].

Definition 3.3.1. Suppose S is any set and $(G, \star)$ is any group. A *(left) group action* is a function $\cdot\colon G \times S \to S$ that satisfies the following axioms:

- for every $s \in S$ we have $1_G \cdot s = s$
- for every $s \in S$ and every $g, h \in G$, we have $g \cdot (h \cdot s) = (g \star h) \cdot s$

A set equipped with a group action by G will be called a *G-set*.

We also include the following useful definition involving G-sets.

Definition 3.3.2. Suppose G is a group and S_1 and S_2 are two sets on which G acts. We say that a function $f\colon S_1 \to S_2$ is a *G-equivariant function of G-sets* if for all $s \in S_1$ and all $g \in G$ we have $f(g \cdot s) = g \cdot f(s)$. In the case where the function f happens to be a bijection, we then say that it is a *G-equivariant bijection of G-sets*.

Another, and indeed equivalent, way we can think of group actions is by group homomorphisms. Indeed, for any set S we may consider the permutation group on S. If we have a homomorphism from a group G to this permutation group on S, then S will become a G-set. We explore this carefully in the following theorem.

Theorem 3.3.1. *Suppose X is a set with corresponding group of permutations $Symm(X) = \{\varphi\colon X \to X \mid \varphi \text{ a permutation}\}$. Assume Γ is a group. If $f\colon \Gamma \to Symm(X)$ is a group homomorphism, then f gives rise to an action by permutations* *on* X *by* $\gamma \cdot x = (f(\gamma))(x)$ *for all* $\gamma \in \Gamma$.

Proof. Assume the above. We aim to show that Γ acts on X as described above. So, we consider the map $\Gamma \times X \to X$ defined by $(\gamma, x) \mapsto (f(\gamma))(x)$, and we denote this by $\gamma \cdot x$. Since f and every φ are well-defined (because f is a group homomorphism and every φ is a permutation), our map is well-defined and hence we may call it a function. So, we want to show that this function satisfies the axioms of group action.

First, consider the identity element 1_Γ of Γ. By definition, for every $x \in X$ we observe that $1_\Gamma \cdot x = (f(1_\Gamma))(x)$. Because $f \colon \Gamma \to \mathrm{Symm}(X)$ is a homomorphism, we know that $f(1_\Gamma) = \mathrm{id}$, the identity permutation on X defined by $\mathrm{id}(x) = x$ for every $x \in X$. So, we observe that $1_\Gamma \cdot x = (f(1_\Gamma))(x) = \mathrm{id}(x) = x$, satisfying the first axiom.

For the second axiom, we consider any $x \in X$ and any $\gamma_1, \gamma_2 \in \Gamma$. Suppose that $f(\gamma_1) = \varphi_1$ and that $f(\gamma_2) = \varphi_2$, for some permutations φ_1, φ_2. Because f is a group homomorphism, we have that $f(\gamma_1\gamma_2) = f(\gamma_1)f(\gamma_2) = \varphi_1 \circ \varphi_2$. Taking this into account, as well as using the definition of function composition, we observe that

$$\begin{aligned}
\gamma_1 \cdot (\gamma_2 \cdot x) &= \gamma_1 \cdot ((f(\gamma_2))(x)) \\
&= \gamma_1 \cdot (\varphi_2(x)) \\
&= (f(\gamma_1))(\varphi_2(x)) \\
&= \varphi_1(\varphi_2(x)) \\
&= (\varphi_1 \circ \varphi_2)(x) \\
&= (f(\gamma_1)f(\gamma_2))(x) \\
&= (f(\gamma_1\gamma_2))(x) \\
&= (\gamma_1\gamma_2) \cdot x
\end{aligned}$$

which shows that the second axiom for group action is satisfied. Indeed, we see that both axioms of group actions are satisfied, and we have that Γ does indeed act on X via Symm(X) as required, completing the proof. ■

In the specific case where a group acts on a metric space, a case we will become quite acquainted with, we have the following corollary:

Theorem 3.3.2. *Suppose $X = (X, d)$ is a metric space, with corresponding group of isometries Isom$(X) = \{\varphi\colon X \to X \mid \varphi$ an isometry$\}$, where we recall that an isometry is a function $f\colon X \to X$ such that for all $x, y \in X$ $d(x, y) = d(f(x), f(y))$. Next, assume Γ is a group. If $f\colon \Gamma \to$ Isom(X) is a group homomorphism, then f gives rise to an action by isometries of Γ on X by $\gamma \cdot x = (f(\gamma))(x)$ for all $\gamma \in \Gamma$.*

Groups acting on metric spaces now introduces a notion of distance that we lacked when we were acting on just a set. This inspires a definition.

Definition 3.3.3. Suppose G is a group and $X = (X, d)$ is a metric space on which G acts. We say that the action of G on X is an *action by isometries* if for all $g \in G$ and for all $x, y \in X$, we have $d(x, y) = d(g \cdot x, g \cdot y)$.

We introduce some new terminology here - while the following definitions are not limited to group actions on metric spaces, we will mainly consider them in this context.

Definition 3.3.4. Let Γ act on a set X with corresponding group homomorphism $\varphi\colon \Gamma \to$ Symm(X). We say that the action of Γ on X is *faithful* if the group homomorphism φ is injective. Equivalently, the action is faithful if for all $x \in X$, $\gamma \cdot x = x$ if and only if $\gamma = 1_\Gamma$ - in other words, a faithful action cannot have fixed points.

Next, we introduce a similar definition.

Definition 3.3.5. Let Γ act on a set X. We say that the action of Γ on X is *free* if $\gamma \cdot x = x$ implies $\gamma = 1_\Gamma$ for any $x \in X$. Equivalently, let the function $\varphi_x \colon \Gamma \to X$ be defined for all $x \in X$ by $\varphi_x(\gamma) = \gamma \cdot x$. We then say that the action of Γ on X is *free* if the function φ_x is injective for every $x \in X$.

In this manner, we can imagine a free action as being "super" faithful. Indeed, freeness implies faithfulness. We prove this here:

Theorem 3.3.3. *Suppose* Γ *acts on* X *a set, with corresponding group homomorphism* $\varphi \colon \Gamma \to \mathit{Symm}(X)$. *If the action of* Γ *on* X *is free, then the action of* Γ *on* X *is faithful.*

Proof. We proceed by contraposition. Suppose the action of Γ on X is not faithful. This simply means that the homomorphism $\varphi \colon \Gamma \to \mathrm{Symm}(X)$ is not injective. Equivalently, $\ker(\varphi) \neq \{1_\Gamma\}$. So, there must exist some $\gamma \in \Gamma \setminus \{1_\Gamma\}$ for which $\varphi(\gamma) = \mathrm{id}$, the identity permutation on X. Let $x \in X$ be arbitrary yet fixed. If $\varphi(\gamma) = \mathrm{id}$, then $x = \mathrm{id}(x) = (\varphi(\gamma))(x) = \gamma \cdot x = \varphi_x(\gamma)$. Now, notice that $\gamma \neq 1_\Gamma$, yet $\varphi_x(\gamma) = x = 1_\Gamma \cdot x = \varphi_x(1_\Gamma)$. This, by definition, makes φ_x not injective, and we conclude that the action of Γ on X is not free; the proof by contraposition is complete. ∎

3.4 Diffuse Actions

Now, we want to talk about diffuseness in a metric sense. This motivates the following definitions. Again, the material in this section is due to Bowditch.

Definition 3.4.1. Suppose Γ is a group that acts on a set X. For any finite subset $A \subset X$, we define the set $\Delta_\Gamma(A)$, which we refer to as the *boundary* of A, by

$$\Delta_\Gamma(A) := \{a \in A \mid \gamma \cdot a \notin A \text{ or } \gamma^{-1} \cdot a \notin A \text{ for all non-trivial } \gamma \in \Gamma\}.$$

Definition 3.4.2. [Bow00] Suppose Γ is a group that acts on any set X. We say that the action of Γ on X is a *diffuse action* if for all finite subsets A, the set $\Delta_\Gamma(A)$ has cardinality greater than 1. We say that the action of Γ on X is a *weakly diffuse action* if for all such subsets A, the set $\Delta_\Gamma(A)$ is nonempty.

This definition warrants a few remarks. First, as groups indeed act on themselves, a diffuse group is precisely a diffuse action of a group on itself (more on this in a moment). We also highlight the fact that, despite having given the definition of X as a set, we will overwhelmingly be taking this set to have the additional structure of a metric space. We begin with the following:

Theorem 3.4.1. *Suppose that G is a group that acts freely on a metric space X. If the action of G on X is diffuse, then the group G itself is diffuse.*

Proof. Assume that the action of G on X is free and diffuse. We proceed by contradiction, and suppose that G is *not* a diffuse group. Invoking Theorem 3.2.6, this means that G is not a weakly diffuse group. By definition of weakly diffuse, this means there is some finite subset $S \subset G$ for which $\Delta(S) = \emptyset$.

Since $\Delta(S)$ is empty, it must be the case that for all $s \in S$, there exists some $1 \neq g_s \in G$ for which both $g_s s \in S$ and $g_s^{-1} s \in S$. Let $x_0 \in X$ be arbitrary yet fixed, and consider the orbit of x_0 in S:

$$\mathrm{Orb}_S(x_0) = \{s \cdot x_0 \mid s \in S\}.$$

Since $\mathrm{Orb}_S(x_0)$ is finite (because S is finite), we consider the set $\Delta(\mathrm{Orb}_S(x_0))$. Let $s_0 \cdot x_0$ in $\mathrm{Orb}_S(x_0)$ be arbitrary yet fixed, and consider the elements $g_{s_0}^{\pm 1}$ in G. We notice that

$$g_{s_0}^{\pm 1} \cdot (s_0 \cdot x_0) = (g_{s_0}^{\pm 1} s_0) \cdot x_0 \in \mathrm{Orb}_S(x_0)$$

because $g_{s_0}^{\pm 1} s_0 \in S$ (from the assumption that $\Delta(S)$ is empty). This means, as $s_0 \cdot x_0$ was taken to be arbitrary in $\mathrm{Orb}_S(x_0)$, that $\Delta(\mathrm{Orb}_S(x_0)) = \emptyset$. We thus have a finite non-empty subset X that has empty boundary. This means the action of G on X is not weakly diffuse, and hence is not diffuse by Theorem 3.2.6, contradicting assumption that the action of G on X is diffuse; we conclude G itself must be diffuse.

■

3.5 Spherical Diffuseness

We have discussed group actions that are themselves diffuse. We take the time now to address what happens when a group acts on a metric space, and how we may derive a form of diffuseness from this.

The definition below gives a name to a condition considered by Bowditch, who did show that it implies diffuseness of the group action.

Definition 3.5.1. Suppose G is a group and (X, d) is a metric space on which G acts. We say that the action of G on X is *spherically diffuse* if for all $g \in G$ and for all $x, y \in X$, if $g \cdot y \neq y$ we have

$$d(x, y) < \max\{d(x, g \cdot y), d(x, g^{-1} \cdot y)\}.$$

We now show that if a group acts spherically diffusely on a metric space, then G itself must be a weakly diffuse group. We remark that the following is an original contribution distinct in content from Bowditch's claims in Lemma 2.1 of [Bow00].

Theorem 3.5.1. *Suppose Γ is a torsion-free group and suppose that $X = (X, d)$ is* *a metric space. If the action of* Γ *on* X *is spherically diffuse, then the action of* Γ *on* X *is weakly diffuse.*

Proof. Suppose Γ acts on $X = (X, d)$ a metric space and that said action is spherically diffuse.

Now, we consider any finite subset $A \subseteq \Gamma$ with $|A| \geq 2$. Assume that $|A| = n \in \mathbb{Z}$, where $n \geq 2$. Assume then that $A = \{a_i\}_{i=1}^n$.

Let $x \in X$ be arbitrary yet fixed. Now, consider the set $A \cdot x \subseteq X$ defined by $A \cdot x = \{a_i \cdot x \mid a_i \in A\}$. Now, let $r \in \mathbb{R}$ be defined by

$$r := \max\{d(x, a_i \cdot x) \mid 1 \leq i \leq n\}.$$

By definition of r, we observe that $A \cdot x \subseteq B_r(x)$, the open ball of radius r centered at x.

Now, we pick some $a_j \in A$ such that $d(x, a_j \cdot x) = r$, and notice that $a_j \cdot x \in S_r(x) = \{y \in X \mid d(x, y) = r\}$.

Since we assume that the action of Γ on X is spherically diffuse, we see that for all $\gamma \in \Gamma$, it holds that $d(x, a_j \cdot x) < \max\{d(x, \gamma a_j \cdot x), d(x, \gamma^{-1} a_j \cdot x)\}$.

We prove that $a_j \in \Delta(A)$. Suppose towards a contradiction that $a_j \notin \Delta(A)$. This means that there would exist some $\tilde{\gamma} \in \Gamma$ such that $\tilde{\gamma} a_j \in A$ and also $\tilde{\gamma}^{-1} a_j \in A$. Since we earlier set $A = \{a_i\}_{i=1}^n$, we set $\tilde{\gamma} a_j = a_k$ and $\tilde{\gamma}^{-1} a_j = a_m$, for some $1 \leq k \leq n$ and $1 \leq m \leq n$. Notice then that $\tilde{\gamma} a_j \cdot x = a_k \cdot x \in A \cdot x$ and that $\tilde{\gamma}^{-1} a_j \cdot x = a_m \cdot x \in A \cdot x$.

Recall that $d(x, a_j \cdot x) = r = \max\{d(x, a_i \cdot x) \mid 1 \leq i \leq n\}$. It thus follows that $d(x, a_j \cdot x) = r \geq d(x, a_k \cdot x)$ and that $d(x, a_j \cdot x) = r \geq d(x, a_m \cdot x)$.

Bearing these inequalities in mind, we return to the spherical diffuseness of Γ's action on X. For all $\gamma \in \Gamma$, we see that $d(x, a_j \cdot x) < \max\{d(x, \gamma a_j \cdot x), d(x, \gamma^{-1} a_j \cdot x)\}$.

Choosing $\gamma = \tilde{\gamma}$, we see

$$\begin{aligned}
r &= d(x, a_j \cdot x) \\
&< \max\{d(x, \tilde{\gamma} a_j \cdot x), d(x, \tilde{\gamma}^{-1} a_j \cdot x)\} \\
&= \max\{d(x, a_k \cdot x), d(x, a_m \cdot x)\} \\
&\leq r
\end{aligned}$$

giving us the contradiction that $r < r$. So, we observe that $a_j \in \Delta(A)$, and hence $\Delta(A)$ is nonempty. By definition, this makes the action of Γ on X weakly diffuse, completing the proof. ■

We mention that this also tells us that a spherically diffuse action of G on a metric space X also tells us that the group itself is diffuse, as we have already showed that weak diffuseness is equivalent to diffuseness.

We are now at a position where we can relate the three forms of diffuseness, and do so in the chain of implications below:

$$\text{Spherical Diffuseness} \implies \text{Weak Diffuseness} \iff \text{Diffuseness.}$$

Chapter 4

CAT(-1) Spaces and Group Actions on Them

In this chapter we will prove the main claim of the thesis: that Bowditch actions of groups on CAT(-1) spaces affirm the Kaplansky conjectures in the group ring made from the group in question and an arbitrary integral domain. To do this, we first need to talk about CAT(κ) spaces in general before talking about properties of CAT(-1) spaces in general. In doing so we will review important details from hyperbolic geometry, which will be invaluable to our arguments. After properly defining Bowditch actions on arbitrary metric spaces, we will prove that a Bowditch action on a CAT(-1) space is a spherically diffuse action.

4.1 Introduction to CAT(κ) Spaces

The first order of business is to familiarize ourselves with the basic notions of general CAT(κ) spaces before dealing specifically with CAT(-1) spaces. Informally speaking, CAT(κ) spaces are ways of generalizing the notions of standard two dimensional Euclidean space $\mathbb{E}^2$. There is an entire branch of mathematics - non-Euclidean geometry - devoted to studying such spaces. While we will briefly discuss some geometry, we will mainly be concerned with these spaces as metric spaces. We specifically talk about the various *model spaces* M^2_κ for κ any real number in two dimensions. We mention that the exposition in this section follows that of Bridson and Haefliger. [BH99]

Definition 4.1.1. For κ any real number, the *model spaces* M^2_κ are defined as follows.

If $\kappa = 0$, then M^2_0 is real Euclidean space $\mathbb{E}^2$ equipped with standard Euclidean metric $d_{\mathbb{E}^2}$.

If $\kappa < 0$, then M^2_κ is real hyperbolic space $\mathbb{H}^2 = \{\mathbf{r} \in \mathbb{R}^2 \mid ||\mathbf{r}|| < 1\}$, with corresponding hyperbolic metric $d_{\mathbb{H}^2}(\mathbf{r}, \mathbf{s}) := \text{arccosh}\left(1 + \dfrac{2||\mathbf{r} - \mathbf{s}||^2}{(1 - ||\mathbf{r}||^2)(1 - ||\mathbf{s}||^2)}\right)$ scaled by a factor of $\dfrac{1}{\sqrt{-\kappa}}$.

If $\kappa > 0$, then M^2_κ is the two-sphere $\mathbb{S}^2 := \{\mathbf{u} \in \mathbb{R}^3 \mid ||\mathbf{u}|| = 1\}$ with corresponding spherical metric $d_{\mathbb{S}^2}(\mathbf{u}, \mathbf{v}) := \arccos(\mathbf{u} \cdot \mathbf{v})$ scaled by a factor of $\dfrac{1}{\sqrt{\kappa}}$.

An important characteristic of these model spaces is that of diameter.

Definition 4.1.2. For a model space M^2_κ, we say that the *diameter* of M^2_κ, denoted by D_κ, will be ∞ if $\kappa \leq 0$, and if $\kappa > 0$ we say that $D_\kappa = \dfrac{\pi}{\sqrt{\kappa}}$

We briefly mention that since we will be working in metric spaces of negative curvature, we may make a specific definition regarding the model space of the hyperbolic plane. We take this definition from Hitchman. [Hit18]

Definition 4.1.3. The *Poincaré disk*, $\mathbb{D}$, is the set of all points $z \in \mathbb{C}$ such that $|z| < 1$. The set $\mathbb{D}$ is called the *hyperbolic plane.*

This is a warm up to the content we will see in Section 4.2, and indeed concretely yields the family of model spaces M^2_κ for $\kappa < 0$.

Now that we have these model spaces in mind, we provide the following definitions.

Unless otherwise stated, $X = (X, d)$ will be an arbitrary metric space.

Definition 4.1.4. We say that a *geodesic segment* joining two points $p, q \in X$ is the image of a path of length $d(p, q)$ joining them. We will denote the geodesic segment joining p and q by $[p, q]$.

Definition 4.1.5. A *geodesic triangle* $\Delta = \Delta(p, q, r)$ in X consists of three points p, q and r, called its *vertices*, and three geodesic segments $[p, q]$, $[q, r]$ and $[r, p]$, called its *sides*, joining them. If a point $x \in X$ lies in $[p, q] \cup [q, r] \cup [r, p]$, we will say that $x \in \Delta(p, q, r)$.

Definition 4.1.6. A *comparison triangle* $\overline{\Delta} = \overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ in $(\mathbb{E}^2, d)$ for the triangle $\Delta = \Delta(p, q, r)$ in (X, d_X) is a Euclidean triangle satisfying

1. $d(\overline{p}, \overline{q}) = d_X(p, q)$
2. $d(\overline{q}, \overline{r}) = d_X(q, r)$
3. $d(\overline{r}, \overline{p}) = d_X(r, p)$

Similarly, a point $\overline{x} \in [\overline{q}, \overline{r}]$ is called a *comparison point* for the point $x \in [q, r]$ if $d_X(q, x) = d(\overline{q}, \overline{x})$. Comparison points on $[\overline{p}, \overline{q}]$ and $[\overline{r}, \overline{p}]$ are defined in a similar way.

It is a natural question then to ask about angles of these various triangles. Unfortunately, measuring angles in a non-Euclidean setting is not so simple. We build up to this new definition of angle as follows.

Definition 4.1.7. For a geodesic triangle $\Delta = \Delta(p, q, r)$ in X and its comparison triangle $\overline{\Delta} = \overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ in $\mathbb{E}^2$, the interior angle of $\overline{\Delta}$ at point $\overline{p}$ is called the *comparison angle* between q and r at p, and is denoted by $\overline{\angle}_p(q, r)$.

We can now define the notion of angle in a non-Euclidean sense.

Definition 4.1.8. For a metric space X and positive real numbers a, a', we let $c \colon [0, a] \to X$ and $c' \colon [0, a'] \to X$ be two geodesic paths such that $c(0) = c'(0)$. Given

$t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\overline{\Delta}(\overline{c(0)}, \overline{c(t)}, \overline{c(t')})$ and the comparison angle $\overline{\angle}_{c(0)}(\overline{c(t)}, \overline{c(t')})$. We define the *Alexandrov angle* between the geodesic paths c and c' to be the number $\angle(c, c') \in [0, \pi]$ given by

$$\begin{aligned}\angle(c, c') &= \limsup_{t,t' \to 0} \overline{\angle}_{c(0)}(\overline{c(t)}, \overline{c'(t')}) \\ &= \lim_{\varepsilon \to 0} \left(\sup_{0<t,t'<\varepsilon} \overline{\angle}_{c(0)}(\overline{c(t)}, \overline{c'(t')}) \right).\end{aligned}$$

With all of these preliminaries out of the way, we are finally able to define a CAT(κ) space.

Definition 4.1.9. Let (X, d_X) be a metric space and let $\kappa \in \mathbb{R}$. Let Δ be a geodesic triangle in X with perimeter less than $2D_\kappa$. Let $\overline{\Delta} \subset M^2_\kappa$ be a comparison triangle for Δ. We say that Δ satisfies the *CAT(κ) inequality* if for all points $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, it holds that $d_X(x, y) \leq d(\overline{x}, \overline{y})$.

If $\kappa \leq 0$, then we say that X is a *CAT(κ) space* if X is a geodesic metric space all of whose geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then we say that X is a *CAT(κ) space* if X is a D_κ-geodesic metric space - which is to say that all pairs of points a distance less than D_κ apart are joined by a geodesic - all of whose geodesic triangles with perimeter less than $2D_\kappa$ satisfy the CAT(κ) inequality.

We give the following theorem regarding CAT(κ) spaces without proof.

Theorem 4.1.1. *Let X be a metric space and let κ, κ' be real numbers.*

1. *If X is a CAT(κ) space, then it is a CAT(κ') space for every $\kappa' \geq \kappa$.*

2. *If X is a CAT(κ') space for all $\kappa' > \kappa$, then X is also a CAT(κ) space.*

4.2 CAT(-1) Spaces: Hyperbolic Geometry and Trees

Now that we know about CAT(κ) spaces at large, we can focus specifically on CAT(-1) spaces. These are spaces whose geodesic triangles are "slimmer" or "as slim" as triangles drawn in $\mathbb{H}^2$, the hyperbolic plane.

Of course, this means that $\mathbb{H}^2$ itself qualifies as a CAT(-1) space. We will be working quite a bit from here on out in hyperbolic geometry, so we take this time to review some basic facts regarding the geometry of this space.

4.2.1 The World of Hyperbolic Geometry

We will not go too in depth with the theory behind hyperbolic geometry, but will briefly touch on key definitions and ideas, omitting any proofs of various formulas. We want to mention that this is exposition of standard background based on Hitchman; Chapter 5 of [Hit18] is an excellent source for the reader wishing to catch up on hyperbolic geometry and see in-depth proof of formulae.

First we recall the hyperbolic equivalents of sine and cosine.

Definition 4.2.1. The *hyperbolic sine function*, denoted $\sinh x$, is defined to be

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

similarly, the *hyperbolic cosine function*, denoted $\cosh x$, is defined to be

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The theorems below are given without proof; we refer the curious reader to [Hit18] for in-depth proofs and derivations of various formulas.

Theorem 4.2.1. *Lines in hyperbolic geometry are formed by either diameters of the Poincaré disk or by semicircles meeting the unit circle at right angles.*

Theorem 4.2.2. *Angles in hyperbolic geometry are defined to be the Alexandrov angles in the metric space* $(\mathbb{H}^2, d_{\mathbb{H}^2})$.

Next we introduce the hyperbolic equivalents of the Law of Cosines and the Law of Sines.

Theorem 4.2.3. *Suppose that a hyperbolic triangle has interior angles* α, β *and* γ *and opposite hyperbolic side lengths* $a, b,$ *and* c, *respectively. Then the following hold:*

1. *(First Hyperbolic Law of Cosines)*

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma).$$

2. *(Hyperbolic Law of Sines)*

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.$$

A specific case happens when we have a right hyperbolic triangle. When we do, we get the analog of the Pythagorean theorem for hyperbolic triangles:

Theorem 4.2.4. *In a right hyperbolic triangle with hypotenuse* c *and legs* a *and* b, *we have that*

$$\cosh(c) = \cosh(a)\cosh(b).$$

4.2.2 Trees

We saw above that a CAT(-1) space can have geodesic triangles that can be at most as "fat" as those comparison triangles in $\mathbb{H}^2$. Hyperbolic space itself thus qualifies as a CAT(-1) space since its geodesic triangles were exactly the same as comparison triangles in the model space.

What if we went to the opposite extreme? Indeed, we clarify just what we mean by a tree with the following definition, for which we cite Bridson and Haefliger. [BH99]

Definition 4.2.2. An $\mathbb{R}$-*tree* is a metric space T with the following properties

- there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in T$
- if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$

In this sense, the trees that we consider are not like the combinatorial objects we meet in graph theory; instead, we think of them as topological realizations of those trees. We may thus think of edges in a tree as closed intervals (usually of unit length), and we may have points in our topological realization that are actually on an edge, hence blurring the lines (pun intended) between "point" and vertex.

A very useful fact about these topological realizations of trees, which we will be informally calling *trees*, is the following, which we cite from [BH99] and give without proof.

Theorem 4.2.5. $\mathbb{R}$-*trees are CAT*(κ) *for all* κ.

The obvious corollary that we glean from this theorem is that trees are CAT(-1), a fact that is invaluable to us. Indeed, with this theorem in mind, we often say that trees are CAT$(-\infty)$, since they are CAT(κ) for every κ, even as κ approaches negative infinity. This is what we mean when we say that trees are the "most" CAT(-1).

4.3 Group Actions on CAT(-1) Spaces

We are now well and familiar with CAT(-1) spaces and the two most extreme types. We now want to investigate what happens when groups act on these spaces, and see how these actions can tell us ring theoretical information. We accomplish this by showing that if a group acts in a special way, to be defined momentarily, on these spaces, then the action must be spherically diffuse. If we can do this, then the Kaplansky Conjectures will follow, as we know how spherical diffuseness leads to the affirmation of the conjectures.

First, we give a useful definition.

Definition 4.3.1. Let G be a group and let (X, d) be a metric space on which G acts by isometries. For any $g \in G$, we say that the *axis of* g, denoted $\text{axis}(g)$, is the set of all elements of X that are minimally displaced by the action of g - in other words,

$$\text{axis}(g) = \{x \in X \mid d(x, g \cdot x) = \min_{y \in X} d(y, g \cdot y)\}.$$

We will refer to the distance $d(x, g \cdot x)$ as the *translation length of* g, which we denote by $\text{tr}(g)$.

This definition will be paramount, because it is key to the claim that the thesis sets out to prove, as we shall soon see. Indeed, we define the following, perhaps the most important definition of the thesis.

Definition 4.3.2. Suppose $X = (X, d)$ is a metric space and G is a group. Suppose $x \in X$ is arbitrary yet fixed. If G acts by isometries on X such that for all non-trivial $g \in G$ we see that $\text{tr}(g) \geq 2\ln(1 + \sqrt{2})$ - that is, $d(x, g \cdot x) \geq 2\ln(1 + \sqrt{2})$ - we will say that the action is a *Bowditch action*.

This definition is inspired by Dr. Brian Bowditch, whose paper [Bow00] was very inspirational to the direction of this project. The number $2\ln(1+\sqrt{2})$ itself is derived from work that Bowditch did in the paper, where he uses it as a lower bound for translation lengths to show that a fundamental group of a certain kind of manifold is diffuse.

Now, a very nice property of Bowditch actions on CAT(-1) spaces is that translation axes are merely lines. Indeed, we will work up towards this result by stating the following definition and thereom. Again, we cite Bridson and Haefliger for this definition. [BH99].

Definition 4.3.3. Let $X = (X, d)$ be a metric space, and f an isometry on X. We say that f is *hyperbolic* if the function $d_f \colon X \to \mathbb{R}$ defined by $x \mapsto d(x, f(x))$ attains a strictly positive minimum.

Bowditch actions are therefore hyperbolic since their strictly positive minima are precisely $2\ln(1+\sqrt{2})$. This is particularly useful because under a hyperbolic isometry, the axis of translation must be nonempty; in other words, if G has a Bowditch action on a CAT(-1) space and $g \in G$, then we will see that $\text{axis}(g) \neq \emptyset$.

Another useful property of these axes deals with convexity, the definition of which we recall below.

Definition 4.3.4. Suppose (X, d) is a geodesic metric space, with $S \subseteq X$ a subset. Given $s_1, s_2 \in S$, we say that S is *convex* if the path $[s_1, s_2]$ lies entirely within S.

Bridson and Haefliger mention in Proposition II.6.2 of [BH99] that if X is a CAT(0) space and G is a group acting by isometries on X, then for all $g \in G$ $\text{axis}(g)$ is a closed convex set. This will be useful momentarily. First, we quote the following theorem from Bridson and Haefliger.

Theorem 4.3.1. *Let X be a CAT(0) space and let f be a hyperbolic isometry of X. The axis of f, the set of all minimally displaced elements under f, is isometric to a product $Y \times \mathbb{R}$ with the L^2 product metric.*

In our situation, this means that if G has a Bowditch action on a CAT(-1) space, which we know is CAT(0) by Theorem 4.1.1, then $\text{axis}(g) \cong Y \times \mathbb{R}$, for $g \in G$ non-trivial and some non-empty set Y. We can claim Y to be nonempty because Bowditch actions are hyperbolic, meaning that we cannot have empty axes.

Consider this set Y. We know that we have an injective isometry $Y \times \mathbb{R} \to X$ whereby $y \times \mathbb{R} \mapsto \text{axis}(g_y)$, for some arbitrary $g_y \in G$. Suppose our Y was not a singleton. Since $\text{axis}(g)$ is convex by Proposition II.6.2 mentioned above, we know it is also connected. If $\text{axis}(g)$ is connected and furthermore isometric to $Y \times \mathbb{R}$, then Y itself must be connected. So, consider $x, y \in Y$, and consider any geodesic path $[x, y]$ in Y. This path is homeomorphic to some closed interval $[0, L]$ by definition of path. So, we see that $[x, y] \times \mathbb{R} \cong [0, L] \times \mathbb{R} \hookrightarrow X$ isometrically. Note that a triangle in $[0, L] \times \mathbb{R}$ is Euclidean, since $[0, L] \times \mathbb{R} \cong \mathbb{E}^2$. Specifically, the angles of any geodesic triangle in this "ribbon" sum to π exactly. However, X is CAT(-1), and so any geodesic triangle in a CAT(-1) space cannot have interior Alexandrov angle measures that sum to π; indeed, the "fattest" CAT(-1) space is hyperbolic space, and hyperbolic triangles are famous for having interior angles summing less than π. In other words, we have a contradiction. Indeed, it must be the case that Y is merely a singleton. The Cartesian product of a singleton with all of $\mathbb{R}$ produces a line. This reasoning stands as a proof for the following theorem.

Theorem 4.3.2. *Suppose X is a CAT(-1) space and G exhibits a Bowditch action on X. For any non-trivial $g \in G$, we see that axis(g) is a line.*

We take the time now to mention a few more useful theorems before diving into specific cases of actions on CAT(-1) spaces. The following theorems are taken from Bridson and Haefliger in [BH99].

Theorem 4.3.3. *Suppose $\kappa \in \mathbb{R}$ and suppose X is a CAT(κ) space. The Alexandrov angle between the sides of any geodesic triangle in X with distinct vertices is no* *greater than the angle between the corresponding sides of its comparison triangle in* M_κ^2.

Since we will be working in CAT(-1) space, this tells us that geodesic triangles in these spaces can be no "fatter" than hyperbolic triangles, whose angle measures are already bounded above by π.

Theorem 4.3.4. *Let* $X = (X, d)$ *be a CAT(0) space and let* C *be a convex subset which is complete in the metric* d*, where we recall that a metric space is complete if and only if every Cauchy sequence of elements in the space converge to a limit that is also in the space.*

- *For every* $x \in X$ *there exists a unique point* $\pi(x) \in X$ *such that* $d(x, \pi(x)) = d(x, C) := \inf_{y \in C} d(x, y)$.
- *If* x' *belongs to the geodesic segment* $[x, \pi(x)]$*, then* $\pi(x') = \pi(x)$.
- *Given* $x \notin C$ *and* $y \in C$*, if* $y \neq \pi(x)$ *then* $\angle(x, \pi(x), y) \geq \dfrac{\pi}{2}$.

In particular, since lines are convex sets, projections from a point in our space onto an arbitrary axis will intersect the axis at least orthogonally. However, since we are working in CAT(-1) spaces and not CAT(0), we have the following useful theorem.

Theorem 4.3.5. *Suppose* (X, d) *is a CAT(-1) space, with arbitrary yet fixed* $x, y \in X$ *and arbitrary yet fixed* $g \in G$*. Denote by* $\overline{x}$ *the element in axis*(g) *that minimizes the distance from* x *to the axis; in other words,* $d(x, \overline{x}) \leq d(x, p)$ *for all* $p \in$ *axis*(g)*. We define* $\overline{y}$ *similarly. We observe that* $d(\overline{x}, \overline{y}) \leq d(x, y)$*. Furthermore, the geodesics joining* x *and* $\overline{x}$ *and* y *and* $\overline{y}$ *intersect axis*(g) *at Alexandrov angles that are at least* $\dfrac{\pi}{2}$.

This works nicely in our favor. Indeed, everything we have seen so far in this chapter will help us in proving the main claim of the thesis. We set out to do so presently.

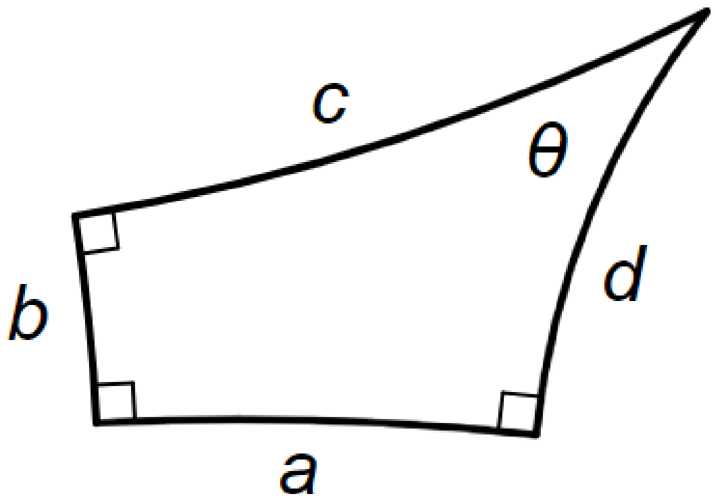

Figure 4.1: A Lambert Quadrilateral

4.4 Bowditch Actions Are Spherically Diffuse

4.4.1 Preamble to the Proof

We will need a preamble before our work begins in earnest, as we will be working with hyperbolic geometry. Specifically, there is a type of hyperbolic shape that will be used frequently in our work, and we define such a figure presently.

Definition 4.4.1. In hyperbolic geometry, a *Lambert quadrilateral* is a four-sided polygon that contains exactly three right angles.

We include a picture of a Lambert quadrilateral for reference, and we refer the reader to Figure 4.1.

We will use properties of Lambert quadrilaterals to our advantage, and state as a theorem the following facts.

Theorem 4.4.1. *Suppose we have a Lambert quadrilateral with side lengths a, b, c and d and angle θ as seen in Figure 4.1. Then the following equalities hold:*

- $\sinh(a)\sinh(b) = \cos(\theta)$

- $\cosh(a) = \cosh(c)\sin(\theta)$
- $\tanh(a)\cosh(b) = \tanh(c)$
- $\sinh(a)\cosh(d) = \sinh(c)$
- $\tanh(a)\sinh(d) = \cot(\theta)$
- $\tanh(c)\tanh(d) = \cos(\theta)$

As an aside, we mention that this means that we need only know two values to determine the other three uniquely.

Now, besides working with these quadrilaterals, we also will be working with hyperbolic trigonometry. These trigonometric functions can be difficult to use, so we may take logarithms to make our calculations easier. We begin presently, and mention that the motivation for these computations is to present a mechanism for incorporating Bowditch's constant of $2\ln(1+\sqrt{2})$.

Note that for any $x \in \mathbb{R}$, we have $x = \ln(e^x) < \ln(e^x + e^{-x})$, and therefore

$$x - \ln(2) < \ln(e^x + e^{-x}) - \ln(2) = \ln\left(\frac{e^x + e^{-x}}{2}\right) = \ln(\cosh(x)).$$

Notice that since $e^{-x} > 0$ for all $x \in \mathbb{R}$, we know that $e^x + e^{-x} > e^x$, meaning that $\ln(e^x + e^{-x}) > x$. Let the error function $\epsilon(x)$ be given by $\epsilon(x) = \ln(e^x + e^{-x}) - x$, and since $x < \ln(e^x + e^{-x})$ for all $x \in \mathbb{R}$ we know that $\epsilon(x) > 0$ for all $x \in \mathbb{R}$.

Note that when we evaluate the error function at Bowditch's constant of $2\ln(1+\sqrt{2})$, we have

$$\begin{aligned}
\epsilon(2\ln(1+\sqrt{2})) &= \ln(e^{2\ln(1+\sqrt{2})} + e^{-2\ln(1+\sqrt{2})}) - 2\ln(1+\sqrt{2}) \\
&= \ln(e^{\ln((1+\sqrt{2})^2)} + e^{\ln((1+\sqrt{2})^{-2})}) - \ln((1+\sqrt{2})^2) \\
&= \ln(e^{\ln(3+2\sqrt{2})} + e^{\ln\left(\frac{1}{3+2\sqrt{2}}\right)}) - \ln(3+2\sqrt{2}) \\
&= \ln\left(3+2\sqrt{2}+\frac{1}{3+2\sqrt{2}}\right) - \ln(3+2\sqrt{2}) \\
&= \ln\left(\frac{3+2\sqrt{2}+\frac{1}{3+2\sqrt{2}}}{3+2\sqrt{2}}\right) \\
&= \ln\left(1+\frac{1}{(3+2\sqrt{2})^2}\right) \\
&= \ln\left(1+\frac{1}{17+12\sqrt{2}}\right) \\
&= \ln(18-12\sqrt{2}) \approx 0.02901.
\end{aligned}$$

Notice that if $\epsilon(x) = \ln(e^x + e^{-x}) - x$, then

$$\begin{aligned}
\epsilon'(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} - 1 \\
&= \frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{e^x + e^{-x}}{e^x + e^{-x}} \\
&= \frac{e^x - e^{-x} - e^x - e^{-x}}{e^x + e^{-x}} \\
&= \frac{-2e^{-x}}{e^x + e^{-x}} \\
&< 0
\end{aligned}$$

for all real x. This tells us that $\epsilon(x)$ is decreasing on $(-\infty, \infty)$.

Next, we aim to find the limit as x approaches ∞ of $\epsilon(x)$. Notice that ϵ is everywhere continuous, and so we proceed as follows.

$$\begin{aligned}\lim_{x\to\infty} e^{\epsilon(x)} &= \lim_{x\to\infty} e^{\ln(e^x+e^{-x})-x}\\ &= \lim_{x\to\infty} e^{\ln(e^x+e^{-x})}e^{-x}\\ &= \lim_{x\to\infty} (e^x+e^{-x})e^{-x}\\ &= \lim_{x\to\infty} 1+e^{-2x}\\ &= 1\end{aligned}$$

telling us that

$$1 = \lim_{x\to\infty} e^{\epsilon(x)} = e^{\lim_{x\to\infty}\epsilon(x)}$$

by continuity, and we conclude that $\lim_{x\to\infty}\epsilon(x)$ must be 0. Since the limit of this error function is 0 as we approach infinity, and since the function is decreasing, this tells us that $\ln(\cosh(x))$ gets arbitrarily close to $x-\ln(2)$ as x increases. Indeed, we saw that $\epsilon(2\ln(1+\sqrt{2}))\approx 0.02$, and the error only gets smaller. We also mention that throughout the remainder of the thesis that we will often abbreviate $2\ln(1+\sqrt{2})$ to T, for the translation length.

Now, since $\epsilon(x)=\ln(e^x+e^{-x})-x$, it follows that $\ln(e^x+e^{-x})=\epsilon(x)+x$. This means that

$$\ln(\cosh(x)) = \ln\left(\frac{e^x+e^{-x}}{2}\right) = \ln(e^x+e^{-x})-\ln(2) = \epsilon(x)+x-\ln(2).$$

These calculations will be paramount in the proof of the following, perhaps the most important theorem in the thesis.

4.4.2 The Main Result and its Proof

Theorem 4.4.2. *Suppose* Γ *is a group and* X *is a CAT(-1) space on which* Γ *acts. If the action of* Γ *on* X *is a Bowditch action, then the action is spherically diffuse.*

Proof. Suppose $X = (X, d_X)$ is any CAT(-1) space. Consider any $x, y \in X$, and consider $d_X(x, y)$. Let Γ be a group that displays a Bowditch action on X. Let $\gamma \in \Gamma$ be some arbitrary yet fixed non-trivial element in Γ, and consider the points $\overline{x}$ and $\overline{y}$ on axis(γ) that minimize the distance from this axis to x and y, respectively. Now, by the triangle inequality,

$$d_X(x, y) \leq d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\overline{y}, y).$$

As an aside, we mention that this inequality becomes an exact equality if X is a tree.

Next, consider $\gamma \cdot y$ in X. Because Γ acts on X by isometries, we know that $d_X(y, \overline{y}) = d_X(\gamma \cdot y, \gamma \cdot \overline{y})$. Furthermore, because $\overline{y}$ lies on axis(γ), we know that $\gamma \cdot \overline{y} = \overline{\gamma \cdot y}$.

We adopt the following notation:

$$d_X(x, \overline{x}) = A$$

$$d_X(\overline{x}, \overline{y}) = D$$

$$d_X(\overline{y}, y) = E = d_X(\gamma \cdot y, \gamma \cdot \overline{y})$$

With this notation, $d_X(x, y) \leq A + D + E$.

Next, consider the geodesic triangles

$$\Delta_1 = \Delta(x, \overline{x}, \overline{\gamma \cdot y}) \text{ and } \Delta_2 = \Delta(x, \gamma \cdot y, \overline{\gamma \cdot y})$$

in X. We consider the comparison triangles

$$\Delta_1' = \Delta(x', \overline{x}', \overline{\gamma \cdot y}') \text{ and } \Delta_2' = \Delta(x', (\gamma \cdot y)', \overline{\gamma \cdot y}')$$

in the model space $(\mathbb{H}^2, d_{\mathbb{H}})$. Simply from the definition of comparison triangle, we know that

$$d_{\mathbb{H}}(x', \overline{x}') = d_X(x, \overline{x}) = A$$
$$d_{\mathbb{H}}(\overline{x}', (\overline{\gamma \cdot y})') = d_X(\overline{x}, \overline{\gamma \cdot y}) = D + 2\ln(1 + \sqrt{2}) = D + T$$
$$d_{\mathbb{H}}((\gamma \cdot y)', (\overline{\gamma \cdot y})') = d_X(\gamma \cdot y, \overline{\gamma \cdot y}) = E.$$

Notice that triangles Δ_1' and Δ_2' share the vertices x' and $\overline{\gamma \cdot y}'$ and the edge between them.

By definition of comparison triangle, we know that $d_X(x, \gamma \cdot y)$ must be precisely equal to $d_{\mathbb{H}}(x', (\gamma \cdot y)')$, since these distances represent side lengths in Δ_2 and Δ_2', respectively. Now, in this four-sided figure we consider the angles $\angle x'\overline{x}'(\overline{\gamma \cdot y})'$ and $\angle \overline{x}'(\overline{\gamma \cdot y})'(\gamma \cdot y)'$. We know from Theorem 4.3.3 that these angles must be greater than or equal to the associated Alexandrov angles $\angle_X(x, \overline{x}, \overline{\gamma \cdot y})$ and $\angle_X(\overline{x}, \overline{\gamma \cdot y}, \gamma \cdot y)$. Also, since X is CAT(-1), we know that projections onto a convex subspace must produce angles greater than or equal to $\frac{\pi}{2}$, telling us that

$$\frac{\pi}{2} \leq \angle_X(x, \overline{x}, \overline{\gamma \cdot y}) \leq \angle_{\mathbb{H}}(x', \overline{x}', (\overline{\gamma \cdot y})')$$

and that

$$\frac{\pi}{2} \leq \angle_X(\overline{x}, \overline{\gamma \cdot y}, \gamma \cdot y) \leq \angle_{\mathbb{H}}(x', \overline{\gamma \cdot y}', (\gamma \cdot y)').$$

We thusly consider the most extreme case, in which $\frac{\pi}{2} = \angle_{\mathbb{H}}(x', \overline{x}', (\overline{\gamma \cdot y})')$ and $\frac{\pi}{2} = \angle_{\mathbb{H}}(x', \overline{\gamma \cdot y}', (\gamma \cdot y)')$ - that is, they are both right angles.

Notice that we now have a four-sided figure in $\mathbb{H}^2$ with two right angles and side lengths $A, D + 2\ln(1+\sqrt{2}) = D + T$ and E. This presents us with several cases, depending on how this figure actually looks. The cases we will consider shortly will be broken up into two broad categories: the "Suspension Bridge" Case(s), in which x' and $\gamma \cdot y'$ lie on the same side of the axis of translation; and the "Bow Tie" Case, in which these two points lie on opposite sides of the axis.

We thusly consider five cases. The first case is to some extent a degenerate case that can be used in either the Suspension Bridge case or the Bow Tie case, as we only use the lengths A and E themselves. The second case is similar, as we assume that one of the points in question lies on the axis itself. It is in the third and fourth cases that we explicitly assume that we are working in the Suspension Bridge case, and it is in the fifth case that we assume we are working in the Bow Tie case.

CASE 1 For the first case, we will suppose that $A + E < 2\ln(1+\sqrt{2})$. Figure 4.2 displaying this case can be referred to. We mention that while that figure specifically uses a Suspension Bridge diagram, this first case also applies to the Bow Tie case.

We use a simple projection argument to see that

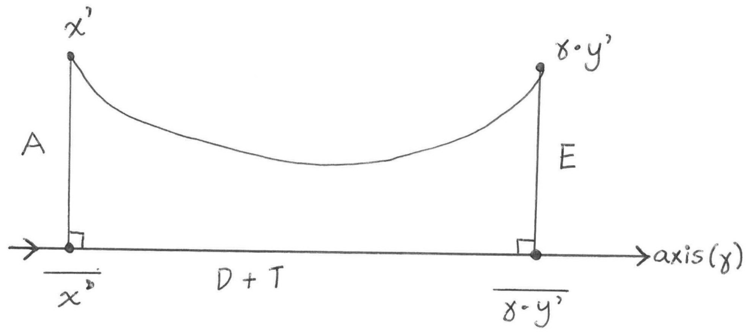

Figure 4.2: Case 1, in which $A + E < 2\ln(1+\sqrt{2})$

$$\begin{aligned}
d_X(x, \gamma\cdot y) &= d_{\mathbb{H}}(x', \gamma\cdot y') \\
&\geq d_{\mathbb{H}}(\overline{x'}, \overline{\gamma\cdot y'}) \text{ by Theorem 4.3.5} \\
&= D + T \\
&= D + 2\ln(1+\sqrt{2}) \\
&> A + D + E \\
&= d_{\mathbb{H}}(x', \overline{x'}) + d_{\mathbb{H}}(\overline{x'}, \overline{y'}) + d_{\mathbb{H}}(\overline{y'}, y') \\
&= d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\overline{y}, y) \\
&\geq d_X(x, y)
\end{aligned}$$

showing spherical diffuseness. This is the easiest of cases, and we now consider a second case.

CASE 2 Here, we assume that $d_{\mathbb{H}}(x', \overline{x'}) = 0$; in other words, x itself lies on the axis of γ. This will mean that $A = 0$. Figure 4.3 shows this case. Now, if $E < 2\ln(1+\sqrt{2})$, then we merely use our work from Case 1. So, we assume that $E \geq 2\ln(1+\sqrt{2})$.

Let $d_{\mathbb{H}}(x', \gamma\cdot y') = H$. We know that, due to the hyperbolic Pythagorean theorem mentioned in Theorem 4.2.4, $\cosh(H) = \cosh(E)\cosh(D+T)$. We aim to avoid

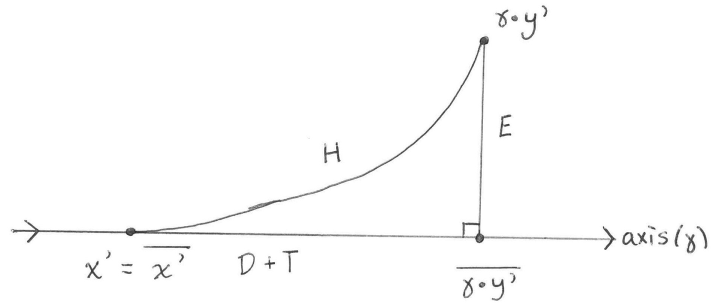

Figure 4.3: Case 2, in which x itself lies on the axis of translation

using arccosh, the inverse hyperbolic cosine function, as much as possible. This is where our preamble comes in. Taking logarithms of both sides yields

$$\ln(\cosh(H)) = \ln(\cosh(E)\cosh(D+T)) = \ln(\cosh(E)) + \ln(\cosh(D+T)).$$

Recalling that $\ln(\cosh(x)) = x - \ln(2) + \epsilon(x)$, we see that

$$H - \ln(2) + \epsilon(H) = D + T - \ln(2) + \epsilon(D+T) + E - \ln(2) + \epsilon(E)$$

and after adding $\ln(2)$ to both sides,

$$H + \epsilon(H) = D + T + E + \epsilon(D+T) + \epsilon(E) - \ln(2).$$

We then see the implication that

$$H = D + T + E + \epsilon(D+T) + \epsilon(E) - \ln(2) - \epsilon(H) > D + T + E - \ln(2) - \epsilon(H)$$

because ϵ is always strictly positive.

Notice that $\cosh(x) \geq 1$ for all real x. Furthermore, since $D + T \geq T$ and since $\cosh(x)$ increases on $(0, \infty)$, we observe that

$$\begin{aligned}\cosh(H) &= \cosh(E)\cosh(D+T)\\ &\geq \cosh(D+T)\\ &\geq \cosh(T)\end{aligned}$$

and so $H \geq T$. Using this realization coupled with the fact that ϵ is decreasing, we have $\epsilon(H) \leq \epsilon(T)$, and furthermore that $-\epsilon(H) \geq -\epsilon(T)$.

Therefore, our earlier inequality becomes

$$H \geq D+T+E-\epsilon(H)-\ln(2) \geq D+T+E-\epsilon(T)-\ln(2).$$

We calculate the value of $T-\epsilon(T)-\ln(2)$ explicitly:

$$\begin{aligned}T-\epsilon(T)-\ln(2) &= 2\ln(1+\sqrt{2})-\ln(18-12\sqrt{2})-\ln(2)\\ &= \ln((1+\sqrt{2})^2)-\ln(36-24\sqrt{2})\\ &= \ln(3+2\sqrt{2})-\ln(36-24\sqrt{2})\\ &= \ln\left(\frac{3+2\sqrt{2}}{36-24\sqrt{2}}\right).\end{aligned}$$

Notice that $\dfrac{3+2\sqrt{2}}{36-24\sqrt{2}} \approx 2.83$ and therefore greater than 1. Since $\dfrac{3+2\sqrt{2}}{36-24\sqrt{2}} > 1$, we conclude that $T-\epsilon(T)-\ln(2) = \ln\left(\dfrac{3+2\sqrt{2}}{36-24\sqrt{2}}\right) > 0$. We now see that

$$H \geq D+T+E-\epsilon(T)-\ln(2) > D+E = 0+D+E = A+D+E$$

Because $H > A + D + E$ we conclude that

$$\begin{aligned}
d_X(x, \gamma \cdot y) &= d_{\mathbb{H}}(x', \gamma \cdot y') \\
&= H \\
&> A + D + E \\
&= d_{\mathbb{H}}(x', \overline{x'}) + d_{\mathbb{H}}(\overline{x'}, \overline{y'}) + d_{\mathbb{H}}(\overline{y'}, y') \\
&= d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\overline{y}, y) \\
&\geq d_X(x, y)
\end{aligned}$$

again showing us that spherical diffuseness is upheld, concluding Case 2.

CASE 3 We aim to build off of Case 2 as follows. For the next two cases, we will still take E to be at least $2\ln(1+\sqrt{2})$, and imagine what happens as A increases. As A, which is to say $d_{\mathbb{H}}(x', \overline{x'})$, increases, there comes a point where $\angle_{\mathbb{H}}(\overline{x'}, x', \gamma \cdot y')$ is a right angle, therefore yielding a Lambert quadrilateral. We use this to our advantage as Lambert quadrilaterals have very nice properties.

Again, denote $d_{\mathbb{H}}(x', \gamma \cdot y')$ by H. Inside the Lambert quadrilateral, draw the geodesic connecting $\overline{x'}$ and $\gamma \cdot y'$ and denote it by H'. For reference, we point the reader to Figure 4.4.

Now, the triangle inequality tells us that $H' \leq A + H$ and therefore $H \geq H' - A$. This will be important momentarily.

Next, the properties of Lambert quadrilaterals mentioned in Theorem 4.4.1 tell us $\cosh(A)\tanh(D + T) = \tanh(H)$. We use this to observe that

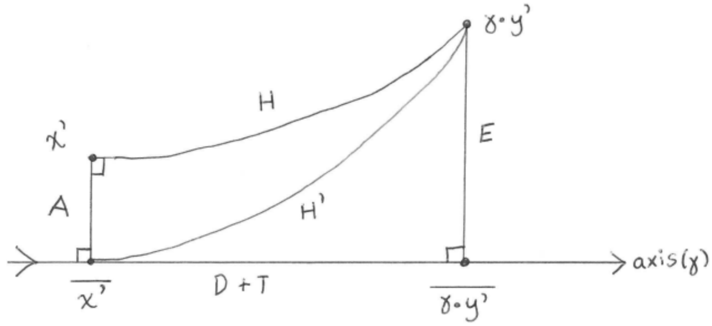

Figure 4.4: Case 3 with a Lambert quadrilateral

$$
\begin{aligned}
\cosh(A) &= \frac{\tanh(H)}{\tanh(D+T)} \\
&< \frac{1}{\tanh(D+T)} \text{ (since } \tanh(x) < 1 \text{ for all real } x) \\
&\leq \frac{1}{\tanh(T)} \text{ (since } \tanh(x) \text{ increases on } (-\infty, \infty)) \\
&= \frac{\cosh(T)}{\sinh(T)} \\
&= \frac{3}{2\sqrt{2}} \\
&= \frac{3\sqrt{2}}{4}.
\end{aligned}
$$

We recall that $\operatorname{arccosh}(x) := \ln(x + \sqrt{x^2 - 1})$ and is defined on all $x \geq 1$. Therefore, we calculate

$$\begin{aligned}
\operatorname{arccosh}\left(\frac{3\sqrt{2}}{4}\right) &= \ln\left(\frac{3\sqrt{2}}{4} + \sqrt{\left(\frac{3\sqrt{2}}{4}\right)^2 - 1}\right) \\
&= \ln\left(\frac{3\sqrt{2}}{4} + \sqrt{\frac{2}{16}}\right) \\
&= \ln\left(\frac{4\sqrt{2}}{4}\right) \\
&= \ln(\sqrt{2}).
\end{aligned}$$

Now, since $\cosh(x)$ and $\operatorname{arccosh}(x)$ are both increasing and injective for $x \geq 1$, we see that

$$\cosh(A) < \frac{3\sqrt{2}}{4} \implies A < \operatorname{arccosh}\left(\frac{3\sqrt{2}}{4}\right) = \ln(\sqrt{2}).$$

We will recall this in a moment, and for now continue with other calculations.

Next, we consider the right triangle with vertices $\overline{x'}, \overline{\gamma \cdot y'}$ and $\gamma \cdot y'$. By the hyperbolic Pythagorean theorem, $\cosh(H') = \cosh(D+T)\cosh(E)$. Taking logarithms, $\ln(\cosh(H')) = \ln(\cosh(D+T)) + \ln(\cosh(E))$. We again use our error function to our advantage, as this equation simplifies to

$$H' - \ln(2) + \epsilon(H') = D + T - \ln(2) + \epsilon(D+T) + E - \ln(2) + \epsilon(E)$$

Furthermore, solving for H' yields

$$\begin{aligned}
H' &= D + T + \epsilon(D+T) + E - \ln(2) + \epsilon(E) - \epsilon(H') \\
&> D + T + E - \ln(2) - \epsilon(H')
\end{aligned}$$

as our error function ϵ is strictly positive. Now, notice that

$$\cosh(H') = \cosh(D+T)\cosh(E) > \cosh(D+T) \geq \cosh(T)$$

because $E \geq T > 1$ by assumption. This tells us that $H' > T$. Since ϵ is decreasing, this means that $\epsilon(H') < \epsilon(T)$ and thus $-\epsilon(H') > -\epsilon(T)$. So,

$$H' > D + T + E - \ln(2) - \epsilon(H') > D + T + E - \ln(2) - \epsilon(T).$$

Earlier, we mentioned that $H \geq H' - A$. So, knowing what we know now,

$$H \geq H' - A > D + T + E - \ln(2) - \epsilon(T) - A.$$

In Case 2, we found the exact value of $T - \ln(2) - \epsilon(T)$ to be $\ln\left(\frac{3+2\sqrt{2}}{36-24\sqrt{2}}\right)$. Taking this into account, along with the recent discovery that $A < \ln(\sqrt{2})$ - and therefore $-A > -\ln(\sqrt{2})$ - we calculate the following:

$$\begin{aligned} T - \ln(2) - \epsilon(T) - A &= \ln\left(\frac{3+2\sqrt{2}}{36-24\sqrt{2}}\right) - A \\ &> \ln\left(\frac{3+2\sqrt{2}}{36-24\sqrt{2}}\right) - \ln(\sqrt{2}) \\ &= \ln\left(\frac{3+2\sqrt{2}}{\sqrt{2}(36-24\sqrt{2})}\right) \\ &= \ln\left(\frac{3+2\sqrt{2}}{36\sqrt{2}-48}\right). \end{aligned}$$

Our next goal is to show that $T - \ln(2) - \epsilon(T) - A > A$. First, we perform the following calculation that will help us in just a moment:

$$
\begin{aligned}
\frac{3+2\sqrt{2}}{72-48\sqrt{2}} &= \left(\frac{3+2\sqrt{2}}{72-48\sqrt{2}}\right)\left(\frac{72+48\sqrt{2}}{72+48\sqrt{2}}\right) \\
&= \frac{216+144\sqrt{2}+144\sqrt{2}+192}{5184-4608} \\
&= \frac{408+288\sqrt{2}}{576} \\
&> \frac{408+288}{576} \\
&= \frac{696}{576} \\
&> 1.
\end{aligned}
$$

We use this calculation to observe that

$$
\ln\left(\frac{3+2\sqrt{2}}{36\sqrt{2}-48}\right) - \ln(\sqrt{2}) = \ln\left(\frac{3+2\sqrt{2}}{\sqrt{2}(36\sqrt{2}-48)}\right) = \ln\left(\frac{3+2\sqrt{2}}{72-48\sqrt{2}}\right) > \ln(1) = 0
$$

implying that $\ln\left(\frac{3+2\sqrt{2}}{36\sqrt{2}-48}\right) > \ln(\sqrt{2})$. At last, we see that

$$
T - \ln(2) - \epsilon(T) - A = \ln\left(\frac{3+2\sqrt{2}}{36\sqrt{2}-48}\right) > \ln(\sqrt{2}) > A.
$$

We return to our original calculation and conclude that

$$
H \geq H' - A > D + E + T - \ln(2) - \epsilon(T) - A > D + E + A.
$$

Thus $H > A + D + E$ and we conclude that

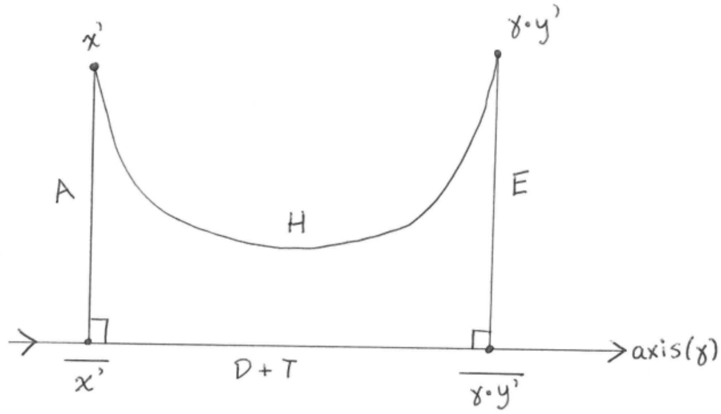

Figure 4.5: The Starting Point for Case 4

$$
\begin{aligned}
d_X(x, \gamma \cdot y) &= d_{\mathbb{H}}(x', \gamma \cdot y') \\
&= H \\
&> A + D + E \\
&= d_{\mathbb{H}}(x', \overline{x'}) + d_{\mathbb{H}}(\overline{x'}, \overline{y'}) + d_{\mathbb{H}}(\overline{y'}, y') \\
&= d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\overline{y}, y) \\
&\geq d_X(x, y)
\end{aligned}
$$

again showing us that spherical diffuseness is upheld, concluding Case 3.

CASE 4 For our Fourth Case, we again aim to build off previous work in Cases 2 and 3 by imagining that A grows in length. Angle $\angle_{\mathbb{H}}(\overline{x'}, x', \gamma \cdot y')$ is now no longer right and is indeed acute. Note that $E \geq 2\ln(1+\sqrt{2})$ remains fixed, and in this case we notice that $A \geq \ln(\sqrt{2})$ - indeed, the assumptions in Case 3 implied that $A < \ln(\sqrt{2})$. Let the geodesic $[x', \gamma \cdot y']$ have distance H, and refer to Figure 4.5 for the start of Case 4.

We again aim to show that $H > A + D + E$, as it is this inequality that will yield

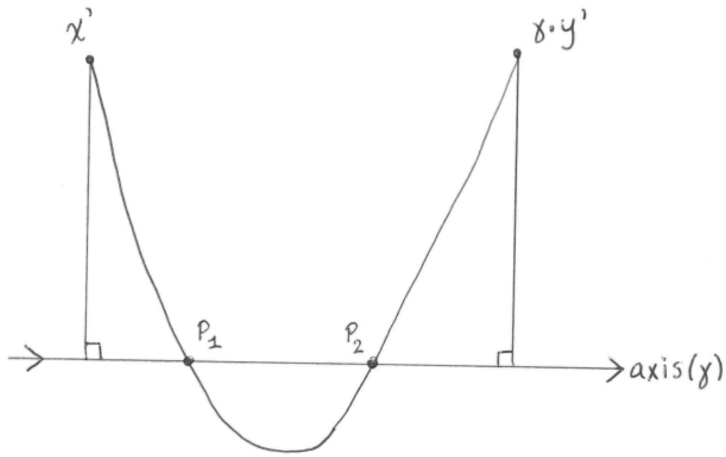

Figure 4.6: A Hypothetical Geodesic Crossing the Axis of Translation

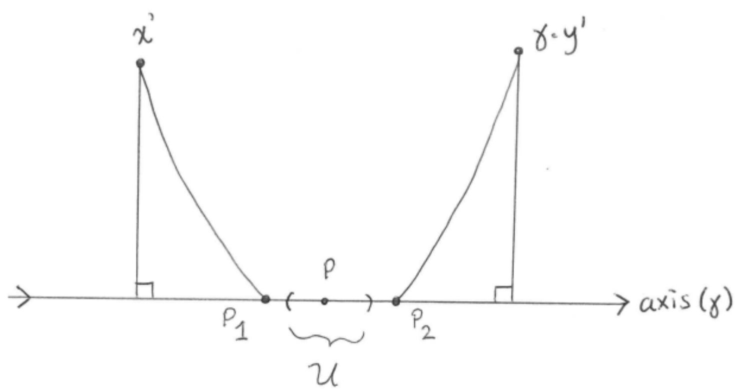

Figure 4.7: A Neighborhood Around Point p

to us spherical diffuseness. The case where $D = 0$ was actually proven by Bowditch, in Section 5 of [Bow00]. So, keeping A and E arbitrary yet fixed, we show that an increase in D maintains spherical diffuseness.

First, let's consider the geodesic $[x', \gamma \cdot y']$ itself. We assert that the geodesic does not cross axis(γ). For if it did, there would be points $p_1, p_2 \in \text{axis}(\gamma) \cap [x', \gamma \cdot y']$, as we may see in Figure 4.6.

Since $\mathbb{H}^2$ is CAT(0), every sub-geodesic is itself a geodesic, and so $[p_1, p_2]$ must be the subpath that lies on axis(γ).

Therefore, with this construction, $[x', \gamma \cdot y'] = [x', p_1] \cup [p_1, p_2] \cup [p_2, \gamma \cdot y']$. However, if

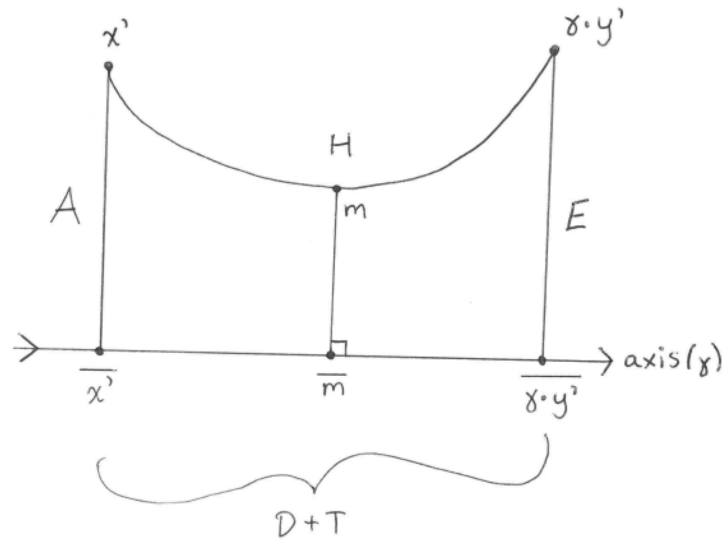

Figure 4.8: Right Angle $\angle_{\mathbb{H}}(m\overline{m}\overline{x'})$

we take some $p \in [p_1, p_2]$ and we take some open neighborhood $\mathcal{U}$ around p, then this neighborhood extends to a geodesic that is unique. We may see such a neighborhood in Figure 4.7. In this case, the geodesic in question will be axis(γ) itself. This implies x' and $\gamma \cdot y'$ lie on axis(γ), a contradiction.

So, the geodesic $[x', \gamma \cdot y']$ must intersect axis(γ) either at a single point or not at all. The case where there is an intersection at a single point will be dealt with in Case 5 - the "Bow Tie Case". So, we assume the geodesic $[x', \gamma \cdot y']$ does not touch axis(γ). Let $m \in [x', \gamma \cdot y']$ be the point that minimizes the distance to the axis. Let $\overline{m}$ be the point on the axis to which m projects. By properties of projections, we know that $\angle_{\mathbb{H}}(m\overline{m}\overline{x'})$ is right. We can see this right angle in Figure 4.8.

Now, we take the projection of $\overline{m}$ onto $[x', \gamma \cdot y']$, and refer to this point on $[x', \gamma \cdot y']$ by $\overline{\overline{m}}$. If $m \neq \overline{\overline{m}}$, then $d_{\mathbb{H}}(\overline{m}, \overline{\overline{m}}) < d_{\mathbb{H}}(\overline{m}, m)$, since $\overline{\overline{m}}$ is the closest point on $[x', \gamma \cdot y']$ to $\overline{m}$. This contradicts the definition of m though. So, m and $\overline{\overline{m}}$ must be one and the same.

We now have a figure that looks like the diagram in Figure 4.9, since m's projection

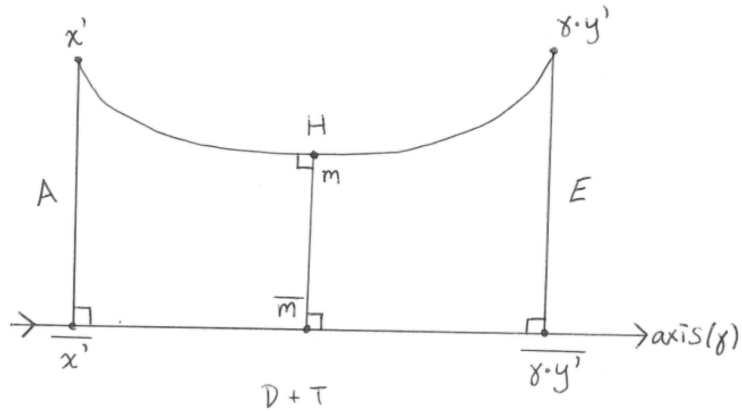

Figure 4.9: Right Angles $\angle_{\mathbb{H}}(m\overline{m}\overline{x'})$ and $\angle_{\mathbb{H}}(x'm\overline{m})$

onto axis(γ) and $\overline{m}$'s projection onto $[x', \gamma \cdot y']$ both form right angles.

We introduce new notation as we have done previously; indeed, refer to the following distances in our diagram as follows:

- $d_{\mathbb{H}}(\overline{x'}, \overline{m}) = D_1$
- $d_{\mathbb{H}}(\overline{m}, \overline{\gamma \cdot y'}) = D_2$
- $d_{\mathbb{H}}(x', m) = H_1$
- $d_{\mathbb{H}}(m, \gamma \cdot y') = H_2$
- $d_{\mathbb{H}}(m, \overline{m}) = \mu$.

A diagram that is labeled with every appropriate measurement is found in Figure 4.10. As an aside, we can draw these pictures in the first place as we are working in $\mathbb{H}^2$.

We draw now a new diagram, where we take some arbitrary yet fixed $D' > D$, where we suppose we have already shown that $H > A + D + E$. If we can then show that $H > A + D' + E$, then we will have spherical diffuseness. Indeed, we have the base case of $D = 0$ from Bowditch. All distances are maintained and new points $\zeta, \overline{\zeta}, p, \overline{p}, q, \overline{q}, \xi$ and $\overline{\xi}$ are named, as well as the new distance D_3. We refer the reader

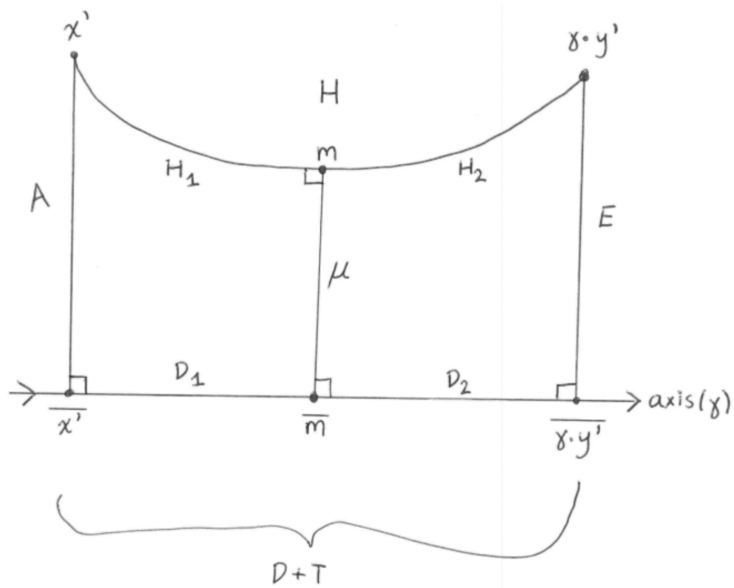

Figure 4.10: Updated Labeling of the Quadrilateral

to Figure 4.11 for a visualization of the new diagram.

Now, we consider the geodesic $[\zeta, \xi]$, noting that it dips below $[\zeta, p]$ and $[\xi, q]$.

Suppose $[\zeta, \xi]$ has length H', and assume this geodesic intersects $[p, \overline{p}]$ and $[q, \overline{q}]$ at points m_1 and m_2, respectively. Let $d_{\mathbb{H}}(\zeta, m_1) = H_1'$, $d_{\mathbb{H}}(\xi, m_2) = H_2'$ and $d_{\mathbb{H}}(m_1, m_2) = \Delta$. With this new length and new points, we refer the reader to Figure 4.12 for an updated diagram.

Indeed, if we were to extend the geodesic segment $[p, \overline{p}]$ (resp. $[q, \overline{q}]$) to a geodesic, then the projection of ζ (resp. ξ) onto this new geodesic would intersect exactly at point p (resp. at point q), since angle $\angle_{\mathbb{H}}(\zeta, p, \overline{p})$ (resp. angle $\angle_{\mathbb{H}}(\xi, q, \overline{q})$) is right. Because m_1 lies on $[p, \overline{p}]$ (resp. m_2 lies on $[q, \overline{q}]$), we know that $d_{\mathbb{H}}(\zeta, m_1) \geq d_{\mathbb{H}}(\zeta, p)$ (resp. that $d_{\mathbb{H}}(\xi, m_2) \geq d_{\mathbb{H}}(\xi, q)$) as projections minimize distances. In other words, $H_1 \leq H_1'$ (resp. $H_2 \leq H_2'$).

Now, we are assuming that $H > A + D + E$. We will be done if we can show that $H' > A + D' + E$. Notice that $H' = H_1' + \Delta + H_2'$, that $\Delta \geq D_3$, and that $D_1 + D_2 = D + T$, the latter of which implies $D_3 = D' - D$.

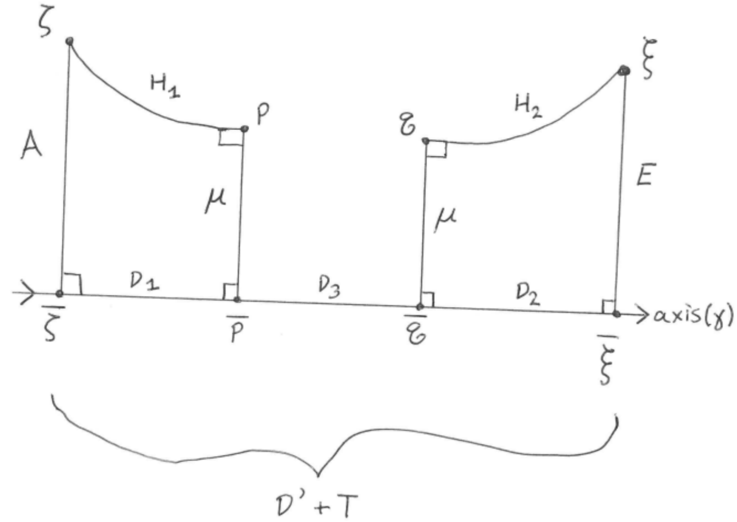

Figure 4.11: New Diagram With New Lengths

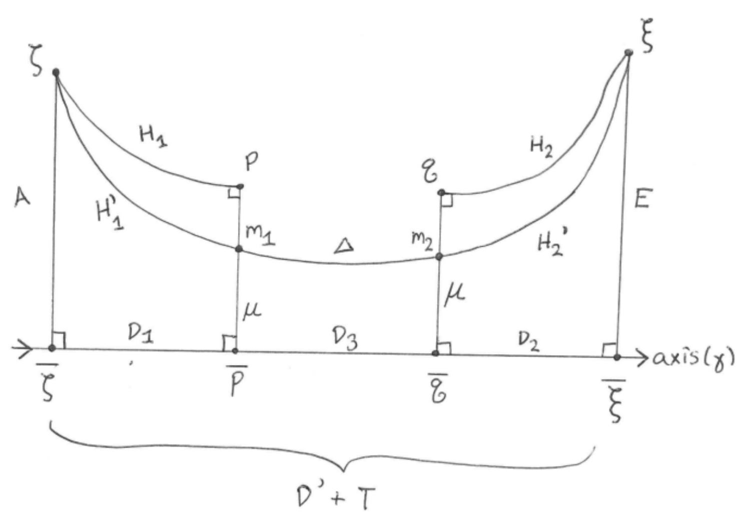

Figure 4.12: Quadrilateral with the Addition of New Points New Lengths

Finally, we observe that

$$
\begin{aligned}
H' &= H_1' + \Delta + H_2' \\
&\geq H_1 + \Delta + H_2 \\
&= H + \Delta \\
&> A + D + E + \Delta \\
&\geq A + D + E + D_3 \\
&= A + D + E + D' - D \\
&= A + D' + E.
\end{aligned}
$$

Indeed, $H > A + D + E$ implies that $H' > A + D' + E$. In other words, keeping A and E constant while increasing D maintains spherical diffuseness. As Bowditch showed spherical diffuseness for $D = 0$, we therefore will have spherical diffuseness for *any* value of D. This concludes Case 4.

CASE 5 We finally turn our attention to the fifth and final case, which we dub the "Bow Tie". In this case, x' and $\gamma \cdot y'$ lie on opposite sides of the axis of translation. In this situation, the geodesic $[x, \gamma \cdot y']$ crosses the axis at some point m on the axis. A diagram of this situation can be found in Figure 4.13.

We consider the point p on the same side of the axis of translation as x' that is the reflection of point $\gamma \cdot y'$ in the axis. In this fashion, $d_{\mathbb{H}}(\gamma \cdot y', m) = d_{\mathbb{H}}(p, m)$ and $d_{\mathbb{H}}(\gamma \cdot y', \overline{\gamma \cdot y'}) = d_{\mathbb{H}}(p, \overline{\gamma \cdot y'})$. Furthermore, we consider the geodesic from $[x', p]$ and compare it to the geodesic $[x', \gamma \cdot y']$. A diagram with this additional point and geodesic is found in Figure 4.14.

But we notice that

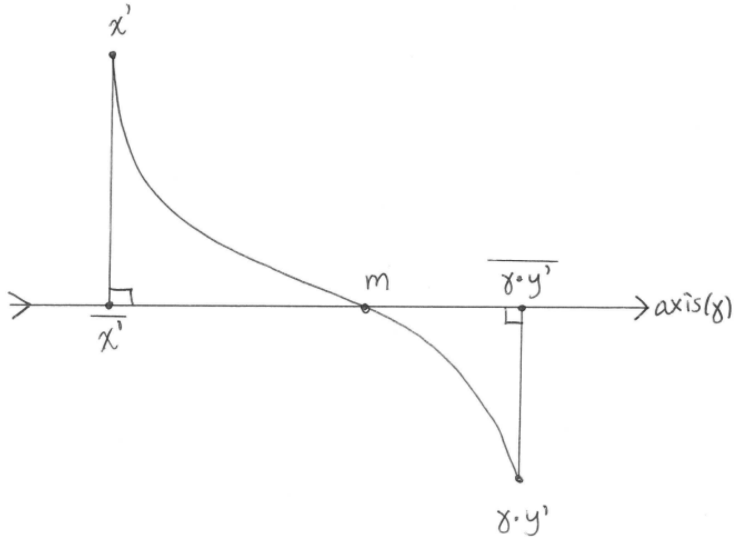

Figure 4.13: Case 5, the Bow Tie Case

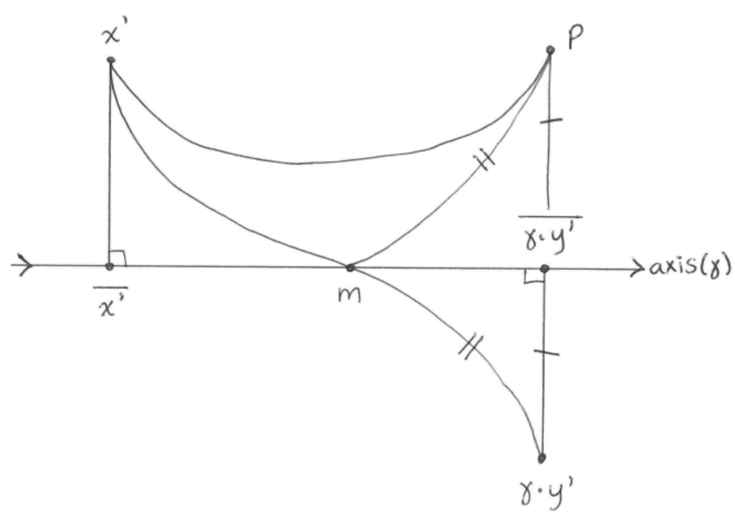

Figure 4.14: Updated Bow Tie

$$\begin{aligned} d_{\mathbb{H}}(x', \gamma \cdot y') &= d_{\mathbb{H}}(x', m) + d_{\mathbb{H}}(m, \gamma \cdot y') \\ &= d_{\mathbb{H}}(x', m) + d_{\mathbb{H}}(m, p) \\ &\geq d_{\mathbb{H}}(x', p) \text{ by the triangle inequality.} \end{aligned}$$

If we therefore find ourselves in the Bow Tie Case, we may as well assume that we are actually in the Suspension Bridge Case, for if spherical diffuseness is shown in the Suspension Bridge case, then the Bow Tie case will follow, since $\gamma \cdot y'$ is further away from x' than p is.

Indeed, for every case we have encountered, we saw that the chain of inequalities worked in our favor. Therefore, we observe that

$$\begin{aligned} d_X(x, y) &\leq d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\overline{y}, y) \\ &= d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\gamma \cdot \overline{y}, \gamma \cdot y) \\ &< d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + d_X(\overline{y}, \gamma \cdot \overline{y}) + d_X(\gamma \cdot \overline{y}, \gamma \cdot y) \\ &= d_X(x, \overline{x}) + d_X(\overline{x}, \overline{y}) + 2 \ln(1 + \sqrt{2}) + d_X(\gamma \cdot \overline{y}, \gamma \cdot y) \\ &= d_X(x, \overline{x}) + d_X(\overline{x}, \gamma \cdot \overline{y}) + d_X(\gamma \cdot \overline{y}, \gamma \cdot y) \\ &= d_{\mathbb{H}}(x', \overline{x}') + d_{\mathbb{H}}(\overline{x}', (\gamma \cdot \overline{y})') + d_{\mathbb{H}}((\gamma \cdot \overline{y})', (\gamma \cdot y)') \\ &< d_{\mathbb{H}}(x', (\gamma \cdot y)') \\ &= d_X(x, \gamma \cdot y) \end{aligned}$$

and indeed we see that

$$d_X(x, y) < d_X(x, \gamma \cdot y) \leq \max\{d_X(x, \gamma \cdot y), d_X(x, \gamma^{-1} \cdot y)\}$$

therefore showing us that this Bowditch action is spherically diffuse, and the proof is complete. ∎

4.5 Tying it All Together

It is at long last that we may state and prove the following central claim of the thesis.

Theorem 4.5.1. *Suppose R is any integral domain, Γ is a torsion-free group, and X is any CAT(-1) space on which Γ acts. If the action of Γ on X is a Bowditch action, then the group ring $R\Gamma$ satisfies the Kaplansky conjectures by lacking non-trivial units, non-trivial zero-divisors and non-trivial idempotents.*

Proof. Suppose R is an integral domain, X is a CAT(-1) space and Γ is a torsion-free group that acts on X under a Bowditch action. By Theorem 4.4.2, we know this action of Γ on X must be spherically diffuse. By Theorem 3.5.1, we know that this action must be weakly diffuse. By Theorem 3.2.6, this makes the action diffuse. By Theorem 3.4.1, this makes Γ a diffuse group itself. By Theorem 3.2.7, Γ must satisfy the two-unique product property. By Theorem 2.1.2, the group ring $R\Gamma$ has no non-trivial units. By Theorem 1.4.5, the group ring $R\Gamma$ has no non-trivial zero divisors. By Theorem 1.4.1, the group ring $R\Gamma$ has no non-trivial idempotents. Indeed, $R\Gamma$ satisfies the Kaplansky conjectures, and we are done. ∎

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