EQUIVARIANT *KK*-THEORY AND ITS APPLICATION IN INDEX THEORY

by

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Dedication

To my father and my grandmother, who did not see this adventure!

Abstract

In this thesis, using the calculation of a couple of Kasparov products of *asymptotically equivariant cycles*, we find the index of an asymptotically equivariant Dirac-Schrödinger operator on a hyperbolic manifold. In fact, using the calculation of the Kasparov products of a couple of asymptotically equivariant cycles, we reduce the problem of finding the index to the case in which the manifold is compact and so the problem reduces to the Atiyah-Singer index theorem.

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Yavar Abdolmaleki

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Table of Notations

$A,\ B$	typical C^*-algebras
$A \otimes C_1$	$A \oplus A$ with odd ordering
$B(H)$	algebra of bounded operators on the Hilbert space H
$C_b(X)$	C^*-algebra of bounded and continuous functions
$CL(V, g)$	Clifford algebra of a quadratic space (V, g)
$C(X, S)$	algebra of continuous functions with values in a a vector bundle
$C^\infty(S)$ or $C^\infty(M; S)$	the space of smooth sections of a vector bundle S
$C_0(X)$	C^*-algebra of continuous functions vanishing at infinity
$D + iV$	Dirac Schrödinger (or Callias type) operator with potential V
$E,\ F$	typical vector bundles or Hilbert modules
$End(S)$	the algebra of endomorphisms of the vector bundle E
$E_p,\ F_p$	typical fibres of a vector bundle
$E \otimes_\phi F$	internal tensor product of E, F
$E \hat{\otimes} F$	external tensor product of E, F
$\Lambda^k T^* M$	k-th exterior algebra of the cotangent bundle
$F_{SO(n)}(M)$	$So(n)$-principal bundle of frame fields of the tangent bundle
$\mathcal{G}(T)$	graph of an operator T
$g(v, v)$	bilinear form or Riemannian metric

$\mathcal{H}$	A typical Hilbert space
$\mathbb{H}^n$	n-dimensional hyperbolic space
$KK(A,B)$	Kasparov KK-groups
$KK_G(A,B)$	equivariant Kasparov KK-groups
$KK_G^{asymp}(A,B)$	asymptotically equivariant KK-groups
$\mathcal{K}_A(E)$	algebra of compact operators on the Hilbert module E
$\mathcal{L}_A(E)$	C^*-algebra of adjointable operators on Hilbert module E
$L^2(E)$	the space of L^2-sections of the vector bundle E
$M,\ N$	typical manifolds
M_n	algebra of complex $n \times n$ matrices
$\mathcal{M}(A)$	multiplier algebra of the C^*-algebra A
$M \times_f N$	warped-product of M and N with warping function f
∇^{LC}	the Levi-Civita connection
∇_e^S	connection on the bundle S
$\Omega(A)$	Gelfand spectrum of the C^*-algebra A
$\Omega^k(M,E)$	space of k-forms with values in E
s	typical section of a vector bundle
$Spin(n)$	spin group of order n

Chapter 1

Introduction

Beyond the earth,

beyond the farthest skies

I try to find Heaven and Hell

Then I hear a solemn voice that says:

"Heaven and Hell are inside."

Khayyam

Kasparov's KK-theory has its roots in the Brown−Douglas−Fillmore theory of the extensions of C^*-algebras [11] and Atiyah and Hirzebruch's cohomology theory, K-theory, which is based on the axiomatization of the properties of elliptic operators on compact manifolds. Not surprisingly so, it is a simultaneous extension of both K-homology and K-theory. One of the interesting problems in the field of differential operators and Functional analysis is to find the index of an operator. In a broad sense, if $T : V \longrightarrow W$ is a linear operator, one wants to find the expression $\dim(\ker(T)) - \dim(\operatorname{coker}(T))$ $-$ when V and W are finite dimensional vector spaces, this expression is nothing but $\dim(V) - \dim(W)$ by the *Rank-nullity* formula that one sees in an introductory course in linear algebra. So the index is not an interesting quantity in the case that spaces are finite dimensional. Operators for which the components of this expression are finite are called *Fredholm* operators. One, in the above mentioned disciplines, often considers a differential operator $D : \Gamma(E) \longrightarrow \Gamma(F)$ on a manifold M. Typical examples of differential operators are the Laplacian, the Dirac operator, and the Dolbeault operator. In this thesis we consider Dirac-Schrödinger operators on spin-$\mathbb{C}$ manifolds$-$ more precisely on hyperbolic manifolds. These are operators of the form $D + V$. Two remarkable facts about the components of this sum are:

i) the Dirac operator D is self-adjoint,

and

ii) If the potential V, a not necessarily scalar perturbation of D, satisfies $V = -V^*$ and VV^* is away from zero outside some compact set, then $D + V$ is Fredholm.

For technical reasons, it is assumed that the manifold is oriented and complete. Completeness ensures that

i) D is essentially self-adjoint and

ii) $D + A$ has a unique closed extension.

Many geometrical and topological quantities can be expressed as integrals on manifolds. The Euler characteristic $\chi(M)$ of a compact Riemannian manifold M, the Gauss−Baunnet theorem and the Riemann−Roch theorem are examples of such representations. A unified and simultaneous extension of the above quantities and theorems is given by the Atiyah−Singer index theorem, which calculates the index of an elliptic operator as an integral. Hence the above examples can be regarded as indices of operators on manifolds.

The anatomy of the thesis can be depicted as follows:

Chapter 2 starts with a short summary on *Hilbert C^*-modules*, the basic structural objects of the KK-theory of Kasparov. This is naturally continued by some facts regarding maps on these spaces. Then *the internal and external tensor product* of Hilbert modules are reviewed; we need these constructions in finding the product module of the Kasparov product of two cycles. Section 2.0.3 is devoted to the concept of the *regularity* of an operator; this is essentially the weakest properties known using which one can build a functional calculus in the realm of unbouded operators. A criteria for checking regularity using the graph of an operator is given.[1]

In section 2.0.4, the tools from differential geometry that we need including connections, Clifford algebras, Spin structure, and Dirac operators and some facts about them, are explained. The final section, 2.0.5, is devoted to the spaces, warped-products and hyperbolic manifolds, on which our operators are defined. Some explicit example of these spaces and the Dirac operator on them is given. [2]

[1]This result is by Woronowicz and a more manageable tool is given in [26].

[2]The reader can find some of the explicit isomorphisms and proofs regarding this part

Chapter 3 starts with the definition of Kasparov's KK-cycles for the bounded and unbounded and equivariant case. A summary of the method by Baaj and Julg, implying that every cycle can be represented by an unbounded cycle is given. The chapter ends with reviewing the interconnection between equivariant cycles and *equivariant exact sequences.*

Chapter 4 is devoted to our new results. Hence, the reader can skip the expository chapters and come back upon them as needed. In this chapter, through the calculation of a couple of equivariant Kasparov products and using homotopy, we calculate the index of equivariant Dirac-Schrödinger operators on hyperbolic manifolds. First we introduce a special compactification of the manifold. Then by introducing a metric on this compactification, we turn it into a Riemannian manifold. Then using the action of an amenable group of isometries on this compactification, we prove the existence of a special invariant state, and use it to build the L^2-spaces of the sections of the manifold and its boundary and show some relations between these two spaces.

The main results of this chapter are: the introduction of $KK_G^{asymp}(A,B)$ for a pair of C^*-algebras A and B using the pulling back of the equivalence relations on the group of the extensions − lemma 4.1.15, which stipulates the conditions under which the potential induces a cycle in $KK_G^{asymp}(\mathbb{C}, C_0(M)\hat{\otimes}C_1)$ − the calculation of the first Kasparov product in proposition 4.1.18 − the calculation of the second Kasparov product in proposition 4.1.20 − the third Kasparov product and homotopy of cycles in proposition 4.1.24, and finally

in the appendix A.0.1.

the main theorem of the chapter, theorem 4.1.25. This theorem calculates the index of a Dirac-Schrödinger operator using the products in the above propositions and the associativity of the product.

Finding the index in a simlar way in the absence of a group action on similar spaces was carried out in [31]. We have extended some of the ideas of that paper, but for the equivariant case we had to create new tools to calculate the index, which now is given by a class in the representation ring of the acting group.

Chapter 2

Exposition

This expository chapter starts with a short summary on *Hilbert modules*, the basic structural objects of the KK-theory of Kasparov; this is continued by graded Hilbert modules and their morphisms. The third section is devoted to the concept of *regularity* of an operator; this is essentially the weakest properties, known by now, using which one can build a functional calculus in the realm of unbounded operators.

In section four, the tools from differential geometry that we need including connections, Clifford algebras, spin structure, and Dirac operators and some facts about them are explained. The final part of this section is devoted to the spaces, warped products and hyperbolic manifolds, on which these objects are defined.

2.0.1 Hilbert modules

Hilbert C^*-modules were first introduced in the work of Kaplansky. His idea was to generalize Hilbert spaces by allowing the inner product to take values in a commutative C^*-algebra rather than the field of complex numbers. We explain here why this idea might be useful: Let A be a commutative C^*-algebra; then using the Gelfand-Naimark theorem, A can be identified with $C(X, \mathbb{C})$, for some locally compact Hausdorff space. We note that if X is a Riemannian manifold or any Euclidean space, one can analyze it by geometric techniques, among which is the study of vector bundles, which is related to our work.

Definition 2.0.1. *[33, chapter 1] Let A be a C^*-algebra(with or without unit). An* ***inner product A-module*** *or* ***pre-Hilbert A-module*** *is a linear space E which is a right A-module(vector space and module structure are compatible in the sense that $\lambda(xa) = (\lambda x)a = x(\lambda a),\ x \in E, a \in A, \lambda \in \mathbb{C}$), together with a map $\langle , \rangle : E \times E \longrightarrow A$ called the inner product, satisfying:*

i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle \quad \forall x, y \in E, \alpha, \beta \in \mathbb{C}$,

ii) $\langle x, ya\rangle = \langle x, y\rangle a \quad \forall x, y \in E, a \in A$,

iii) $\langle y, x\rangle = \langle x, y\rangle^* \quad \forall x, y \in E$,

iv) $\langle x, x\rangle \geq 0; \quad if\ \langle x, x\rangle = 0\ , then\ x = 0$.

By setting $||x|| := \sqrt{||\langle x, x\rangle||}$, we turn E into a normed space (if we drop the second condition in (iv), $||.||$ will be a semi-norm). A ***Hilbert A-module*** *is a pre-Hilbert A-module which is complete with respect to this norm.*

A vector bundle E over X can be described as follows: Take a Euclidean space H, and for each $t \in X$, let H_t be a subspace of H. Let E be the set of all continuous functions ξ from X into the Hilbert space H, such that $\xi(t) \in H_t, \forall t \in X$. E is naturally endowed with a $C(X)$-valued inner product:

$$\langle \xi, \eta \rangle (t) := \langle \xi(t), \eta(t) \rangle_H.$$

E has the structure of a Hilbert module: given that $\xi \in E$ and $f \in C(X)$, ξf and the inner product is defined pointwise. Since vector bundles are effective tools in the study of manifolds, it is natural to want to extend the above construction to general compact Hausdorff spaces. This is a prototypical example of a Hilbert $C(X)$-module. We say this formally:

Example 2.0.2. *[40, chapter 1] Let X be a compact Hausdorff space and $\mathcal{H}$ a Hilbert space. Let*

$$C(X, \mathcal{H}) := \{f : X \longrightarrow \mathcal{H} : f \text{ is continuous}\}.$$

$C(X, \mathcal{H})$ is a Hilbert $C(X)$-module with $(g.f)(x) := g(x)f(x)$ and $\langle f, g \rangle_{C(X)}(x) := \langle f(x), g(x) \rangle_{\mathcal{H}}$.

Suppose $x \in X$ and we have chosen a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$. Then $S = \{f \in C(X, \mathcal{H}), f(x) \in \mathcal{H}_0\}$ is a Hilbert $C(X)$- submodule of $C(X, \mathcal{H})$.

Of course the axiom of E being a complex vector space is redundant when A is unital. It is actually redundant even in the non-unital case [50].

Definition 2.0.3. *Let E be the space $\oplus_1^\infty B$ of sequences in B that are zero after finitely many terms. Define $\langle .,. \rangle$ on E by*

$$\langle (a_1, a_2, a_3, ...), (b_1, b_2, b_3) \rangle = \sum a_n^* b_n.$$

Then E is a pre-Hilbert B-module and the Hilbert B-module which we obtain by completing it will be denoted H_B.

Example 2.0.4. *[30, Appendix] Suppose X is a locally compact Hausdorff space. A Hilbert $C_0(X)$-module H is a pre-Hilbert $C_0(X)$-module that is complete in the norm on H given by $||e|| := ||\langle e, e \rangle||_\infty^{1/2}$.*

Example 2.0.5. *[30, Appendix] The algebraic tensor product $C_0(X) \odot_{\mathbb{C}} \ell_2$, where $\ell_2 = H_{\mathbb{C}}$, is a pre-Hilbert module with the inner product given by $(f \odot s\ , f' \odot s') := f'f \langle s\ , s' \rangle_{\ell_2}$.*

Example 2.0.6. *[30, Appendix] The Hilbert module $H_{C_0(X)}$ is the completion of $C_0(X) \odot_{\mathbb{C}} \ell_2$ in the above norm, in other words, $H_{C_0(X)}$ is the space of sequences (f_n) such that $\sum f_n f_n^*$ is convergent. The module $H_{C_0(X)}$ has a natural evaluation map with values in ℓ_2, defined as follows:*

For each $y \in X$, let $\pi_y : C_0(X) \odot_{\mathbb{C}} \ell_2 \longrightarrow \ell_2$ be the linear map acting on simple elements by $f \otimes s \mapsto f(y)s$.

We note that on any finite sum of elements, the above map is norm-decreasing. Therefore we may extend this map to the completion of $C_0(X) \odot_{\mathbb{C}}$

ℓ_2, i.e $H_{C_0(X)}$. We note that π_y has the following property:

$\langle f \otimes s\ , g \otimes p\rangle|_y = \langle \pi_y(f \otimes s)\ , \pi_y(g \otimes p)\rangle$, for all simple elements $f \otimes s,\ and\ g \otimes p$. So, this holds for general elements of $H_{C_0(X)}$. Hence:

$\langle a, b\rangle = \langle \pi_y(a), \pi_y(b)\rangle$.

Definition 2.0.7. *[24, chapter 1] Suppose E, F are two Hilbert A-modules. A function $T : E \longrightarrow F$ is adjointable if there is a function denoted $T^* : F \longrightarrow E$ such that*

$$\langle T(e), f\rangle_A = \langle e, T^* f\rangle_A; \quad \forall e \in E,\ \forall f \in F.$$

It can be shown that every adjointable map $T : E \longrightarrow F$ is a bounded linear A-module map.

One of the differences between operators on Hilbert modules and the more familiar ones on Hilbert spaces is that adjoints do not automatically exist. As an example, let $A = C([0, 1])$ and $J = \{f \in A : f(0) = 0\}$ [40, chapter 1]. Then A and J are Hilbert A-modules. Let $E = A \oplus J$, and define $T : E \longrightarrow E$ by $T(f, g) = (g, 0)$. Then T is bounded A-linear with $||T|| = 1$. If T had an adjoint, and $T^*(1, 0) = (f, g)$, then for all $(h, k) \in E$

$$\bar{k} = \langle T(h, k), (1, 0)\rangle_A = \langle (h, k), (f, g)\rangle = \bar{h}f + \bar{k}g.$$

This forces $f = 0$ and $g = 1$, which contradicts $g(0) = 0$. Thus T cannot be adjointable.

Definition 2.0.8. *[24, chapter 1] If E and F are Hilbert A-modules, then* *we denote by $\mathcal{L}_A(E, F)$ the set of adjointable operators from E to F. We write $\mathcal{L}(E)$, or $\mathcal{L}_A(E)$ for emphasis, for $\mathcal{L}(E, E)$.*

One can show that $\mathcal{L}_A(E)$ is a C^*-algebra.

Definition 2.0.9. *[24, chapter 1] A C^*-algebra A can be regarded as a Hilbert A-module by setting $\langle a, b\rangle := a^*b$. Set $\mathcal{M}(A) := \mathcal{L}_A(A)$. $\mathcal{M}(A)$ is called the multiplier algebra of A.*

There are other ways of introducing the multiplier of an algebra (for example as the algebra of double centralizers), but because we are essentially working with Hilbert modules, we use the above identification. However, it is unique up to isomorphism.

2.0.2 Graded Hilbert modules

We quote some definitions and facts about graded algebras form [9, section 14.1].

Let A be a C^*-algebra. A grading automorphism of A is an automorphism α of A, such that $\alpha^2 = 1$ (α is sometimes referred to as the grading automorphism). We define $A_0 := \{a \in A : \alpha(a) = a\}$ and $A_1 := \{a \in A : \alpha(a) = -a\}$. Therefore A_0 and A_1 are closed, linear self-adjoint subspaces of A and satisfy $A_iA_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$. We have $A = A_0 \oplus A_1$ as a Banach space. Note that A_0 is a C^*-subalgebra of A. Elements of A_0 are called *even* and elements of A_1 are called *odd*. To calculate A_0 and A_1, it is helpful to note that

$$A_0 := \left\{ \frac{a+\alpha(a)}{2} : \ a \in A \right\} \ \text{ and } \ A_1 := \left\{ \frac{a-\alpha(a)}{2} : \ a \in A \right\}. \qquad (2.1)$$

If $a \in A_i$ then we say that a is homogeneous of degree i and write $\partial a = i$. If α is the identity map on α then $A_0 = A$ and $A_1 = 0$. The resulting grading is called the trivial grading.

Given a graded C^*-algebra (A, α_A) and homogeneous elements $a, b \in A$, the graded commutator $[a,b]^{gr}$ is defined as $[a,b]^{gr} = ab - (-1)^{\partial a \partial b} ba$. This formula extends to arbitrary a and b by bi-linearity. In particular, if $a \in A_1$, then

$$[a,b]^{gr} = ab - \alpha_A(b)a.$$

Let B be a graded C^*-algebra. A graded (right-) Hilbert $B-$module is a Hilbert $B-$module E together with a decomposition of E as a direct sum of two closed subspaces, E_0 and E_1, compatible with the grading of B in the sense that $X_i B_j \subset E_{i+j}$ and $< E_i, E_j >\subset B_{i+j}$ (The graded components E_i need not be Hilbert sub-modules).

Some standard gradings:

i) The standard odd grading on $A \oplus A$ is given by decomposition $A^{(0)} = \{(a,a) : a \in A\}$ and $A^{(1)} = \{(a,-a) : a \in A\}$.

ii) The standard even grading on $A \oplus A$ is given by decomposition $A^{(0)} = \{(a,0) : a \in A\}$ and $A^{(1)} = \{(0,a) : a \in A\}$.

iii) If A is any (ungraded) C^*-algebra, there is a grading on $M_2(A)$

with $M_2(A)^{(0)}$ the diagonal matrices, and $M_2(A)^{(1)}$ the matrices with zero diagonal. This is an even grading with grading $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The reader can find a short summary on grading structure in [23, Appendix].

If there is a self-adjoint unitary $g \in \mathcal{M}(A)$ with $A^{(n)} = \{a \in A : g^*ag = (-1)^n a\}$, then the grading is called even and g is called a generator operator for the grading. If $A^{(1)} = 0$, the grading is trivial. A trivial grading is even with grading operator 1. A homomorphism $\phi : A \longrightarrow B$ of graded C^*-algebras is a graded homomorphism if $\phi(A^{(n)}) \subset B^{(n)}$ for $n = 0, 1$.

Internal tensor Product [24, section 1.2]

Let E be a Hilbert B-module, and F a Hilbert A-module. Let $\phi : B \longrightarrow \mathcal{L}_A(F)$ be a $*$-homomorphism. Then ϕ makes F into a left B-module: $bx := \phi(b)x, b \in B, x \in F$. Thus we can construct the algebraic tensor product $E \otimes_B F$ which is a right Hilbert A-module in an obvious way: $(x \otimes_B y)a := x \otimes ya$. Now we define

$$\langle ., . \rangle : E \otimes_B F \times E \otimes_B F \longrightarrow A$$

$$\langle x_1 \otimes_B x_2, y_1 \otimes y_2 \rangle := \langle x_2, \phi(\langle x_1, y_1 \rangle) y_2 \rangle; \quad \forall x_1, y_1 \in E,\ x_2, y_2 \in F.$$

Set $N_{EF} := \{z \in E \otimes_B F; \langle z, z \rangle = 0\}$. N_{EF} is an A-submodule. So we can construct the quotient space $E \otimes_B F/N_{EF}$, and the quotient map $q : E \otimes_B F \longrightarrow E \otimes_B F/N_{EF}$. $q(x)a := q(xa), x \in E \otimes_B F, a \in A$ defines a right A-module structure on $E\otimes_B F/N_{EF}$. One can define an A-valued inner product on $E\otimes_B F/N_{EF}$ by $\langle q(x), q(y)\rangle := \langle x, y\rangle, \forall x, y \in E\otimes_B F$. This makes $E\otimes_B F/N_{EF}$ into a pre-Hilbert A-module. The completed Hilbert A-module is denoted by $E\otimes_\phi F$, and the image of $x\otimes_B y$ in $E\otimes_\phi F$ is denoted by $x\otimes_\phi y$. We note that there is a $*$-homomorphism $j : \mathcal{L}_B(E) \longrightarrow \mathcal{L}_A(E\otimes_\phi F)$ defined by $j(m)(x\otimes_\phi y) : m(x)\otimes_\phi y;\ \ \forall x \in E, y \in F, m \in \mathcal{L}_B(E)$. Sometimes the operator j is denoted by $m\otimes_\phi \mathrm{id}$, as it is more suggestive.

External tensor product [24, section 1.2]

Let E be a Hilbert B-module, and F a Hilbert C-module. Then the algebraic tensor product $E\otimes_{\mathbb{C}} F$ is a right module over $B\otimes_{\mathbb{C}} C$ in an obvious way:

$$(e\otimes_{\mathbb{C}} f)(b\otimes_{\mathbb{C}} c) := eb\otimes_{\mathbb{C}} fc\ ;\ \ \forall e \in E, f \in F, b \in B, c \in C.$$

Now we define

$$\langle .\ ,\ .\rangle : E\otimes_{\mathbb{C}} F \times E\otimes_{\mathbb{C}} F \longrightarrow B\otimes_{\mathbb{C}} C$$

$$\langle e\otimes_{\mathbb{C}} f\ ,\ e_1\otimes_{\mathbb{C}} f_1\rangle := \langle e, e_1\rangle \otimes \langle f, f_1\rangle\ ;\ \forall e, e_1 \in E, f, f_1 \in F.$$

Consider $B\otimes_{\mathbb{C}} C$-submodule $N_{EF} := \{z \in E\otimes_{\mathbb{C}} F; \langle z, z\rangle = 0\}$. Let q be the quotient map $q : E\otimes_{\mathbb{C}} F \longrightarrow E\otimes_{\mathbb{C}} F/N_{EF}$. Then $E\otimes_{\mathbb{C}} F$ is equipped with a $B \otimes_{\mathbb{C}} C$-valued inner product defined by

$$\langle q(x), q(y)\rangle := \langle x, y\rangle, \forall x, y \in E \otimes_{\mathbb{C}} F.$$

Let $E \hat{\otimes} F$ denote the completion of $E \otimes_{\mathbb{C}} F/N_{EF}$ in the norm coming from the above inner product.

$E \otimes_{\mathbb{C}} F/N_{EF}$ is a right $B \otimes_{\mathbb{C}} C$-module:

$$q(x)b \otimes_{\mathbb{C}} c = q(x(b \otimes_{\mathbb{C}} c)); \quad x \in E \otimes_{\mathbb{C}} F, b \in B, c \in C$$

We note that $||zb|| \leq ||z||||b||, \forall z \in E \otimes_{\mathbb{C}} F/N_{EF}, b \in B \otimes_{\mathbb{C}} C$. So we can extend the $B \otimes_{\mathbb{C}} C$- module structure in two steps to obtain a right $B \otimes C$-module structure on $E \hat{\otimes} F$. In this way $E \hat{\otimes} F$ becomes a $B \otimes C$-module.

2.0.3 Regularity

In the various contexts in which Hilbert C^*-modules arise, one also needs to study *unbounded adjointable operators*, or what are now known as *regular operators*. These were first introduced by Baaj and Julg in [1], where they gave a nice construction of Kasparov bimodules in KK-theory. Later they were rediscovered by Woronowicz. In this subsection we review these special types of operators. [33, chapter 9] [1] Let H_1, H_2 be Hilbert spaces and $T : \mathfrak{Dom} T \longrightarrow H_2$ be a densely defined linear operator, i.e. $\mathfrak{Dom} T$ is a dense linear subspace of H_1.

[1] Another good source for this part is [32]

Let $\mathfrak{Dom}T^*$ be the space of all $y \in H_2$ such that $x \longmapsto (Tx, y)_2$ defines a continuous linear functional on $\mathfrak{Dom}T$. Since $\mathfrak{Dom}T$ is dense in H_1, there exists a uniquely determined element $T^*y \in H_2$ such that $(Tx, y)_2 = (x, T^*y)_1$ (Riesz representation theorem).

Definition 2.0.10. *The map $y \longmapsto T^*y$ is linear and $T^* : \mathfrak{Dom}T^* \longrightarrow H_1$ is called the adjoint operator to T.*

The lack of a projection theorem in Hilbert $\mathbb{C}^*$-modules causes the theory of unbounded operators to be complicated. To have a reasonable theory (e.g. with a functional calculus for selfadjoint operators), one has to introduce the additional axiom of regularity:

Definition 2.0.11. *A closed densely defined operator T in E is called* regular *if*

i) $\mathfrak{Dom}T^*$ *is dense in E;*

ii) $1 + T^*T$ *has dense image in E.*

We note that in the above definition if E is a Hilbert space, $A = \mathbb{C}$, then conditions $i), ii)$ can be deduced from the closedness of T.

Next, we will state two fundamental facts about the graph $\mathcal{G}(T)$ of a densely defined operator T on Hilbert modules.

Theorem 2.0.12. *[33, chapter 9] Let E, F be two Hilbert A-modules, T be a regular operator from $\mathfrak{Dom}T \subseteq E$ to F, $\mathcal{G}(T)$ be the graph of T and let*

$\mu \in \mathcal{L}(E \oplus F, F \oplus E)$ be the unitary operator given by $\mu(x,y) := (y, -x)$. Then

$$E \oplus F = \mathcal{G}(T) \oplus \mu\mathcal{G}(T^*). \tag{2.2}$$

Proposition 2.0.13. *[33, chapter 9] Suppose $T : \mathfrak{Dom}T \longrightarrow F$ is a closed densely defined operator. Suppose also that T^* is densely defined, and that $\mathcal{G} \oplus \mu\mathcal{G}(T^*) = E \oplus F$. Then T is regular.*

Proof. The projection p along the graph $\mathcal{G}(T)$ and perpendicular to $\mu\mathcal{G}(T^*)$ is a positive element of $\mathcal{L}(E \oplus F)$ and hence has a matrix representation of the form

$$\begin{pmatrix} a & b^* \\ b & d \end{pmatrix},$$

where $0 \leq a \leq 1$ in $\mathcal{L}(E)$, $b \in \mathcal{L}(E,F)$, and $0 \leq d \leq 1$ in $\mathcal{L}(F)$. Since $ran(p) = \mathcal{G}(T)$, and $ran(1-p) = \mu\mathcal{G}(T^*)$, it follows that

$$ran(a) \subseteq \mathfrak{Dom}T, \quad \textit{with } b = Ta,$$

$$ran(b) \subseteq \mathfrak{Dom}T^*, \quad \textit{with } 1 - a = T^*b. \blacksquare$$

Corollary 2.0.14. *[33, chapter 9] Let $T : E \longrightarrow F$ be a closed operator, and suppose that both T, T^* are densely defined. Then T is regular if and only if T^* is regular.*

Proof. Since $\mu^2 = -1$ and the graph $\mathcal{G}(T)$ is invariant under multiplication by the scalar -1, the condition $E \oplus F = \mathcal{G}(T) \oplus \mu\mathcal{G}(T^*)$ is symmetric in T and T^*. Now, by the above theorem and proposition this condition is equivalent to regularity of T. ■

Definition 2.0.15. *[33, chapter 9] An operator T in E is said to be symmetric if $(Tx, y) = (x, Ty)$ whenever $x \in \mathfrak{Dom}T$ and $y \in \mathfrak{Dom}T$.*

The densely defined symmetric operators are thus exactly those that satisfy $\mathcal{G}(T) \subset \mathcal{G}(T^*)$, and the inclusion here means that T^* is an extension of T. If $T = T^*$, then T is said to be self-adjoint.

Lemma 2.0.16. *[33, chapter 9] If $T : E \longrightarrow F$ is symmetric, closed and densely defined, then $T + i$ is injective and has closed range. (The same statement is true for $T - i$).*

Proof. For all x in $\mathfrak{Dom}T$,

$$\begin{aligned}
|(T+i)x|^2 &= \langle Tx + ix, Tx + ix \rangle \\
&= \langle Tx, Tx \rangle - i\langle x, Tx \rangle + i\langle Tx, x \rangle + \langle x, x \rangle \quad \text{(T is symmetric)} \\
&= |Tx|^2 + |x|^2 \\
&= |(x, Tx)|^2.
\end{aligned}$$

These equalities show that the map $(x, Tx) \longmapsto (T+i)x$ is an isometry from $\mathcal{G}(T)$ onto the range of $T + i$. As T is closed, so is the range of $T + i$. The above equalities show that $T + i$ is injective. ■

Lemma 2.0.17. *[33, chapter 9] Suppose T is a self-adjoint densely defined operator in E. Then T is regular if and only if the operators $T \pm i$ are surjective.*

Proof. Suppose T is regular, then $1 + T^2$ has dense range. In fact this operator is surjective. Thus $(T+i)(T-i)$ is surjective. Hence, so is $T+i$. for the same reason $(T-i)$ is surjective. Conversely, as composition of surjective maps are surjective, $1 + T^2$ is surjective, and so T is regular. ■

Definition 2.0.18. *[42, section 2.2] Suppose T is a closed operator on Hilbert space $\mathcal{H}$. A complex number λ belongs to the resolvent set $\rho(T)$ of T if the operator $T - \lambda I$ has a bounded inverse, $(T - \lambda I)^{-1}$, everywhere defined on $\mathcal{H}$ called the resolvent of T at λ, and denoted by $R_\lambda(T)$.The spectrum of T is defined to be the complement of $\rho(T)$ i.e. $\mathbb{C} \setminus \rho(T)$.*

We note that formally the above definition could also be used to define the spectrum for a not necessarily closed operator T, but if $\lambda \in \rho(T)$, then the bounded everywhere defined operator $(T-\lambda I)^{-1}$ is closed, so is its inverse and hence T. Therefore if T is not closed , we would always have $\rho(T) = \varnothing$, so the notion of spectrum becomes trivial. For this reason, it is assumed that T is closed.

2.0.4 Preliminaries from differential geometry

On Spin manifolds and Dirac operators. In order to define Spin manifolds and Dirac operators, we need some knowledge from differential geometry. In this sub-section, we review basic results and definitions needed for this goal.

Definition 2.0.19. *[48, section 27.1] Let E, M, and F be smooth manifolds. Let $\mathfrak{U}$ denote an open cover $\{U_\alpha\}$ of M. A* ***local trivialization*** *with fiber F for a smooth surjection $\pi : E \longrightarrow M$ is an open cover $\mathfrak{U} = \{U\}$ for M together with a collection $\{\phi_U : \pi^{-1}(U) \longrightarrow U \times F | U \in \mathfrak{U}\}$ of fiber-preserving diffeomorphisms $\phi_U : \pi^{-1}(U) \longrightarrow U \times F$:*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & U & \end{array}$$

where η is the projection to the first factor. A ***fiber bundle*** *with fiber F is a smooth surjection $\pi : E \longrightarrow M$ having a local trivialization with fiber F. Suppose G is a Lie group. A smooth fiber bundle $\pi : P \longrightarrow M$ with fiber G is a smooth* ***principal G-bundle*** *if G acts smoothly and freely on P on the right, and the fiber-preserving local trivializations*

$$\phi_U : \pi^{-1}(U) \longrightarrow U \times G$$

are G-equivariant, where G acts on $U \times G$ on the right by

$$(x, h).g = (x, hg).$$

Definition 2.0.20. *[23, chapter 10] Let M be a smooth manifold and S be a smooth complex vector bundle over M. Let $C^\infty(M;S)$ be the space of smooth sections of S. A first order linear differential operator on S is a complex linear map*

$$D : C^\infty(M;S) \longrightarrow C^\infty(M;S)$$

which has the following properties:

i) if s_1 and s_2 are smooth sections of S which agree on an open set $U \subset M$, then Ds_1 and Ds_2 agree on U, and

ii) for each coordinate patch $U \subset M$, if we choose coordinates x_j in U and a trivialization for the bundle S over U, then D can be represented in local coordinates by a formula

$$Du = \sum_j A_j \frac{\partial u}{\partial x_j} + Bu,$$

where A_j and B are smooth, matrix valued functions on U.

The functions A_j and B which appear in the definition depend on the particular coordinate system and trivialization chosen. But if $\xi = \sum_j \xi_j dx_j$ is a cotangent vector at a point $x \in M$, and if we form the expression

$$\sigma_D(x,\xi) := \sum_j A_j \xi_j,$$

then $\sigma_D(x,\xi)$, interpreted as an endomorphism of the vector space S_x, is independent of the choice of coordinate.

Definition 2.0.21. *Let* $D : C^\infty(M; S) \longrightarrow C^\infty(M; S)$ *be a differential operator. The* ***symbol*** *of* D *is the vector bundle morphism*

$$\sigma_D : T^*M \longrightarrow End(S)$$

which is defined by the formula above.

Definition 2.0.22. *[23] Let* M *be a smooth manifold and let* S *be a smooth complex vector bundle over* M*. A differential operator* $D : C^\infty(M; S) \longrightarrow C^\infty(M; S)$ *is* ***elliptic*** *if its symbol* $\sigma_D(x, \xi)$ *is an invertible endomorphism of* S_x*, for all non-zero* $\xi \in T^*_x M$*. If* U *is an open subset of* M*, then the operator* D *is elliptic over* U *if the restriction of* D *to* U *is elliptic.*

The above definitions can be extended to the case of two different complex smooth vector bundles. In the following we borrow some standard constructions from [23].

Suppose M is a closed Riemannian manifold.[2] Let E_0, E_1 be two smooth vector bundles on M and $C^\infty(E_i)$ denotes the space smooth sections of E_i, $i = 1, 2$.

Let $D : C^\infty(E_0) \longrightarrow C^\infty(E_1)$ be an elliptic pseudo-differential operator. Choose a smooth measure μ on M and a C^∞ Hermitian structure on E_0, E_1. This means that each fibre E_{ip} is endowed with an inner product turning it into a Hilbert space, and these inner products parameterized by the manifold M are varying in smooth fashion. Each $C^\infty(E_i)$ has then an inner product

[2]Closed means compact without boundary.

given by

$$(s, s') := \int_M \langle s(x), s'(x) \rangle_i d\mu, \ \forall s, s' \in C^\infty(E_i). \tag{2.3}$$

Completion of $C^\infty(E_i)$ with respect to the corresponding inner product is the Hilbert space $L^2(E_i)$.

The operator D can then be viewed as an unbounded operator from $L^2(E_0)$ to $L^2(E_1)$,

$$D : L^2(E_0) \longrightarrow L^2(E_1).$$

We note that this possibly unbounded operator is closable.

Theorem 2.0.23. *[6] If the operator $D : C^\infty(E_0) \longrightarrow C^\infty(E_1)$ is an elliptic pseudo-differential operator on a closed manifold M, then $\ker(D)$ and $\operatorname{coker}(D)$ are finite-dimensional vector spaces, and $\operatorname{Index}(D) = \dim_{\mathbb{C}} \ker(D) - \dim_{\mathbb{C}} \operatorname{coker}(D)$ is well defined.*

2.0.5 Dirac operators

Dirac operators came to exist when Paul Dirac was working on the quantum theory of the electron and tried to factorize the Klein-Gordon equation. The reader can see this in the fundamental paper by Paul Dirac [16] or chapter 10 of [35]. Using Pauli matrices, Dirac factorized the equation. The equation cannot be factored over the field of complex numbers.

Definition 2.0.24. *[34, chapter2,3] The **Clifford algebra** $CL(V, g)$ of a quadratic space (V, g),denoted by $CL(V, q)$, is an algebra generated by the* *vectors $v \in V$ subject to the relations $uv + vu = -2g(u,v)1$ for $u, v \in V$.*

The existence of this algebra can be seen in different ways. The approach that fits our work is the construction of the (real)Clifford algebra as a subalgebra of the endomorphisms of the exterior algebra $End_{\mathbb{R}}(\wedge^* V)$, where $\wedge^* V$ is the exterior algebra of V. To do so, define $\varepsilon(v)$ and $\iota(v)$ by

$$\varepsilon(v) : u_1 \wedge u_2 ... \wedge u_k \mapsto v \wedge u_1 \wedge ... \wedge u_k, \; and$$

$$\iota(v) : u_1 \wedge u_2 ... \wedge u_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} g(v, u_j) \wedge u_1 \wedge ... \wedge \hat{u}_j ... \wedge u_k.$$

Let $c(v) := \varepsilon(v) + \iota(v)$. Then $\varepsilon^2(v) = 0, \iota^2(v) = 0, \; and \; \varepsilon(v)\iota(u) + \iota(v)\varepsilon(v) = g(u,v)1$. Thus:

$$c(v)^2 = g(v,v)1, \; for \; all \; v \in V,$$

$$c(u)c(v) + c(v)c(u) = 2g(u,v)1, \; for \; all \; u, v \in V.$$

Thus these operators on $\wedge^* V$ do provide a representation of $CL(v, g)$. It can be shown that $c(e_{k_1})c(e_{k_2})...c(e_{k_r}) : 1 \mapsto e_{k_1} \wedge_{k_2} ... \wedge e_{k_1} \in \wedge^* V$. We show $e_{k_1} \wedge_{k_2} ... \wedge e_{k_r}$ by e_K in which K is the ordered set $\{k_1 < k_2 < ... < k_r\}$.

The so-called **symbol map**

$$\sigma : CL(V, g) \longrightarrow \wedge V, \; a \mapsto a(1)$$

is inverted by

$$Q: u_1 \wedge u_2 ... \wedge u_r \mapsto \frac{1}{r!} \sum_{\tau \in S_r} c(u_{\tau(1)}) c(u_{\tau(2)}) ... c(u_{\tau(r)}).$$

Usually under this realization of $CL(V,g)$ $c(u)c(v)$ is shown by uv.

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$.

If $E \longrightarrow M$ is a vector bundle, we let $\Omega^k(M,E)$ denote the space of sections of the vector bundle $\Lambda^k T^* M \otimes E$. If F is a vector space then $\Omega^k(M,F)$ denotes the F-valued k-forms on M.

Definition 2.0.25. *[34, chapter2,3] A connection on a vector bundle $\pi : E \longrightarrow M$ is a map $\nabla : C^\infty(M,E) \longrightarrow \Omega^1(M,E)$ satisfying the following property:*

$$\nabla(fs) = df \otimes s + f \nabla s \quad for\ all f \in C^\infty(M), s \in C^\infty(M;E).$$

Let (M^n, g) be a complete Riemannian manifold. Let $CL(M)$ be the Clifford bundle of algebras induced by the tangent bundle TM and the Riemannian metric g. Using the canonical embedding $TM \longrightarrow CL(M)$, the Riemannian metric and the $Levi-Civita$ connection are extended from TM to $CL(M)$. The connection ∇^{LC} on $CL(M)$ preserves the metric and acts as a derivation. A generalized Dirac bundle is a bundle $S \longrightarrow M$ furnished with a left module structure over $CL(M)$, a Hermitian metric $<,>$, and a metric connection ∇^S_e such that i) $\nabla^S_e(\phi \circ s) = \nabla^{LC}_e(\phi) \circ s + \phi \circ \nabla^S_e(s)$; $e \in C^\infty(TM), \phi \in C^\infty(CL(M))$, $s \in C^\infty(S)$.

ii) The action on S by unit vectors in $TM \subset CL(M)$ is a pointwise isometry.

In order for M to be a spin manifold we lift the principal $SO(n)$-bundle defined by the fiber bundle $P_{SO}(M)$ to a principal $Spin(n)$-bundle $P_{Spin}(M)$. In other words, we have the following definition.

Definition 2.0.26. *[34, chapter3 and appendix] A triple (P_{Spin}, π_P, F_P) is a spin structure on the principal bundle $\pi : F_{SO(n)}(M) \longrightarrow M$ when*

i) $\pi_P : P_{Spin} \longrightarrow M$ is a principal $Spin(n)$-bundle over M,

ii) $F_P : P_{Spin} \longrightarrow P_{SO(n)}(M)$ is an equivariant two-fold covering map such that

$$\pi \circ F_P = \pi_P \text{ and } F_P(pq) = F_P\mu(q)$$

and

$$\mu : Spin_n \longrightarrow SO(M) \text{ is the spin representation.}$$

Definition 2.0.27. *[34] Let E be an oriented Riemannian vector bundle with a spin structure $\xi : P_{Spin} \longrightarrow P_{SO}$. Let $\mu : Spin_n \longrightarrow SO(M)$ be the representation given by left multiplication of elements of $Spin_n \in CL^0(\mathbb{R})$.*

A real spinor bundle of E is a bundle of the form

$$S(E) := P_{Spin(E)} \times_{\mu} M,$$

in which M is a left $CL(\mathbb{R}^n)$. In the same way, a complex spinor bundle of is defined to be

$$S_{\mathbb{C}}(E) := P_{Spin}(E) \times M_{\mathbb{C}}.$$

The group $SO(n;\mathbb{R})$ is not simply connected[3]. For $n \geq 3$, its universal covering, $Spin(n)$, is compact and covers $SO(n;\mathbb{R})$ twice. Let $\Delta_n := \mathbb{C}^{2^k} = \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2 \quad for \quad n = 2k, 2k+1$. Elements of Δ_n are called complex spinors. There is a representation $\epsilon : Spin(n) \longrightarrow GL(\Delta_n)$ of the spin group which is compatible with the Clifford multiplication. Consider now those special Riemannian manifolds M^n, called spin manifolds, the frame bundle of which allows a reduction to the double cover $Spin(n)$. One can define the vector bundle S associated with this reduction via the representation $\epsilon : Spin(n) \longrightarrow GL(\Delta_n)$,the so called spinor bundle of the manifold. Then spinor fields over M^n are sections of the bundle S, and as the Euclidean case, the Dirac operator can be introduced by the formula

$$D\psi = \Sigma_{i=1}^{n} e_i.\nabla_{e_i}\psi.$$

Here ∇ denotes the covariant derivative corresponding to the $Levi-Civita$

[3] The reader is referred to [20] for more explanation.

connection of the Riemannian manifold.

Definition 2.0.28. *[34] Let S be a Dirac bundle. A* ***Generalized Dirac operator*** *is the composition of the following*

$$C^{\infty}(S) \longrightarrow C^{\infty}(TM \otimes S) \longrightarrow C^{\infty}(T^{*}M \otimes S) \longrightarrow C^{\infty}(S)$$

in which the first arrow is given by the covariant derivative, the second one by Riemannian metric, and the third arrow is given by the Clifford multiplication.

Locally it can be expressed as

$$D^{S} = D \equiv \sum_{i=1}^{n} e_i \circ \nabla^{S}_{e_i}.$$

This expression is independent of the choice of the local frame $e_1, e_2, ..., e_n$.

Theorem 2.0.29. *Let*

$$(s_1, s_2) := \int_{\Omega} \langle s_1, s_2 \rangle d\ vol, \quad s_1, s_2 \in C^{\infty}(\Omega; S). \tag{2.4}$$

The Dirac operator D is formally self adjoint with this inner product, $(Ds_1, s_2) = (s_1, Ds_2)$, when $s_1,\ s_2 \in C^{\infty}(S)$, and one of them has compact support.

Definition 2.0.30. *[1, section1] Let S be any Dirac bundle. An operator $L : C^{\infty}(S) \longrightarrow C^{\infty}(S)$ is called a perturbed Dirac operator if $L = D + A$ where*

D is the Dirac operator associated to S and A is a zero order differential operator on S which means $A \in C^\infty(End(S))$.[4]

Now suppose $f \in C^\infty(M)$ and for $s_1, s_2 \in C^\infty(S)$ let the vector field V_{s_1,s_2} be defined by $\langle V_{s_1,s_2}, X\rangle = \langle X \circ s_1, s_2\rangle$. Then we have:

Proposition 2.0.31. *[1, 1.13] The following statements are equivalent:*

i) L is a perturbed Dirac operator,

ii) $D(fs) = grad\ f \circ s + fDs$,

iii) the formal adjoint $L^\dagger$ is a perturbed Dirac operator,

iv) $\langle Ls_1, s_2\rangle = \langle s_1, Ls_2\rangle + div V_{s_1,s_2}$.

Let $E \longrightarrow M$ be a G-equivariant Hermitian vector bundle over M, with a G-equivariant Hermitian connection ∇^E on E.

The G-action on E induces an action $U^G : G \times \Gamma_0(E) \longrightarrow \Gamma_0(E)$ defined by

$$\forall g \in G,\ \forall s \in C(M, E),\ \forall x \in M,\ (U^G(s))(x) := g.s(g^{-1}.x).$$

This turns $\Gamma_0(E)$ into a G-equivariant $C_0(M)$-module.

Definition 2.0.32. *[37] Let M and N be two semi-Riemannian manifolds with corresponding metrics $\mathbf{g_M}$ and $\mathbf{g_N}$ and $\pi_M : M \times N \longrightarrow M$ and $\pi_N : M \times N \longrightarrow N$ be the projection on the first and second spaces respectively. Let f be a positive smooth map on M, $f : M \longrightarrow \mathbb{R}^+$. The warped-product*

[4] The reader is also referred to [34] as a comprehensive reference.

$M \times_f N$ is the product manifold $M \times N$ furnished with the metric tensor

$$\mathbf{g} = \pi_{\mathbf{M}}^*(\mathbf{g_M}) + (f \circ \pi_M)^2 \pi_{\mathbf{N}}^*(\mathbf{g_N}).$$

More explicitly, if x is a tangent vector to $M \times N$ at (p, q), then

$$\langle x, x \rangle = \langle d\pi_M(x), d\pi_M(x) \rangle + f^2(p)(d\pi_N(x), d\pi_N(x)).$$

It is standard to use the same notation $\langle , \rangle$ for the inner product of the warped product and the first space but $(,)$ for the second one. M is called the base of $M \times_f N$, and N the fiber, and f the warping function. We note that ***fibers*** $\pi_M^{-1}(p) = p \times N$, and ***leaves*** $\pi_N^{-1}(q) = M \times q$ are semi-Riemannian sub-manifolds of the product. We note that

- $\pi_M(M \times q)$ is an isometry onto $M, \forall q \in N$,
- $\pi_M(p \times N)$ is a homothety onto $N, \forall p \in M$,
- ***fibers*** $p \times N$ and ***leaves*** $M \times q$ are orthogonal at (p, q).

Definition 2.0.33. *[1] A Riemannian manifold M is said to have a warped end if there is a compact set $K \subset\subset M$, and a warped product W(above definition) such that $M \setminus K$ and W are isometric as Riemannian manifolds.*

We make this precise.

Definition 2.0.34. *A topological manifold M is a* ***warped-cone*** *over N if* *there exists a co-compact subset U of M and a homeomorphism*

$$k : U \longrightarrow (0,1) \times N. \tag{2.5}$$

Example 2.0.35. $\mathbb{R}^2 \setminus 0$ *is a warped cone.*[5]

As another example of a warped-cone we have the following:

Example 2.0.36. $\mathbb{R}^3 \setminus 0$ *is a warped cone.* [6]

And the last example is the hyperbolic disk:

Example 2.0.37. $\mathbb{D} := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ *with the hyperbolic metric* $\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$ *has a warped cone structure.* [7]

Example 2.0.38. *We just proved that $\mathbb{R}^2 \setminus 0$ is a warped-cone. The Dirac operator on this space is given by*[8]

$$\begin{pmatrix} 0 & i\partial_x + \partial_y \\ -i\partial_x - \partial_y & 0 \end{pmatrix}. \tag{2.6}$$

Example 2.0.39. *As another example, [36, 11.1.4], let E be a vector bundle over a Riemann manifold (M, g), $C^\infty(M)$ the space of smooth sections of E, and $D : C^\infty(M) \longrightarrow C^\infty(M)$ a Dirac operator. Denote by $\widehat{M}$ the cylinder*

[5]See the Appendix A.0.1 for a proof.

[6]See the Appendix A.0.2 for a proof. This proofs are straight forward and they are just based on commutativity of push-back and differential forms and tensor products.

[7]See the Appendix A.0.4 for the calculation of this one too.

[8]Calculation is given in the Appendix A.1

$\mathbb{R} \times M$, and $\hat{g} = dt^2 + g$ the corresponding metric on $\widehat{M}$. Let $\pi : \widehat{M} \longrightarrow M, (t, x) \mapsto x$ be the projection on the second factor and $\widehat{E} := \pi^* E$ be the pullback of E via π. A section $\hat{u}$ of $\widehat{E}$ is then a smooth 1-parameter family of sections $u(t)$ of E, $t \in \mathbb{R}$. In particular, we can define unambiguously $\partial_t \hat{u}(t_0, x_0) = \lim_{h \to 0} \frac{1}{h}(u(t_0 + h, x_0) - u(t_0, x_0))$, in which the limit is in E_{x_0} which is independent of t. Now let

$$\widehat{D} : C^\infty(\widehat{E} \oplus \widehat{E}) \longrightarrow C^\infty(\widehat{E} \oplus \widehat{E})$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\partial_t + D \\ \partial_t + D & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}. \tag{2.7}$$

*$\widehat{D}$ is a Dirac operator called the **suspension** of D.*

We know that there is a realization of two dimensional hyperbolic space $\mathbb{H}^2$ as the upper half-plane $\mathbb{C}_+$[9] with Poincaré metric in which the action of $SL(2, \mathbb{R})$ by linear transformations

$$SL(2, \mathbb{R}) \times \mathbb{C}_+ \longrightarrow \mathbb{C}_+$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z \right) \mapsto \frac{b + zd}{a + zc} \tag{2.8}$$

gives rise to a group of transformations of $\mathbb{C}_+$ that preserve the hyperbolic metric.

[9]The reader may find the following sources useful [41],[38] [21]

Theorem 2.0.40. *[21, Chapter 5 theorem 3.5] The standard Dirac operator on $\mathbb{H}^n$ is defined by*

$$D = \sum_{i=1}^{n-1} e_i(y\frac{\partial}{\partial x_i} + \frac{1}{2}d\tau(e_ie_n)) + e_ny\frac{\partial}{\partial y}. \tag{2.9}$$

We recall that $\tau : Spin(n) \longrightarrow \mathcal{L}(\mathfrak{H})$ is a representation of $Spin(n)$ on a $\mathfrak{A}_n$-module $\mathfrak{H}$ ($\mathfrak{A}_n$ is the universal Clifford algebra for $\mathbb{R}^n$ with Euclidean inner product), e_1, e_2 are standard orthonormal bases for $\mathbb{R}^2$, and $d\tau : \mathfrak{spin} \longrightarrow \mathcal{L}(\mathfrak{H})$ is the differential of τ.

In case $n = 2$ we get

$$D = e_1(y\frac{\partial}{\partial x} + \frac{1}{2}d\tau(e_1e_2)) + e_2y\frac{\partial}{\partial y}. \tag{2.10}$$

We note that as the quadratic space is indefinite, $Spin(n, \mathbb{R})$ is not connected and we use the following fact:

Theorem 2.0.41. *[21, Theorem 6.15] When (V, Q) is a real indefinite quadratic space, there exists a two-fold covering homomorphism*

$$\sigma : Spin_0(V, Q) \longrightarrow SO_0(V, Q) \tag{2.11}$$

$$\sigma(g)(v) := gvg^{-1}.$$

($Spin_0(V, Q)$ is the connected component of $Spin(V, Q)$ containing the identity.)

There is an action of $SL(2,\mathbb{R}) = Spin_0(2,1)$ on $\mathbb{H}^2$ by fractional linear transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto z. \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{b+dz}{a+cz}$.

It is known that this (connected) group of transformation preserves the hyperbolic metric. We note that the inner product on $\mathbb{R}^{p,q}$ is not definite, so instead of group $Spin(p,q)$, group $Spin_0(p,q)$ is used.

$SL(2,\mathbb{Z})$ acts isometrically on $\mathbb{H}^2$.

The modular group is the subgroup $SL(2,\mathbb{Z})$ of the matrix group $SL(2,\mathbb{R})$ consisting of matrices with integer entries and determinant 1. The group $SL(2,\mathbb{Z})$ lies discretely in $SL(2,\mathbb{R})$. It is one of the most basic examples of a discrete non-abelian group. Two particular elements in $SL(2,\mathbb{Z})$ are

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad and \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It can be shown that S, and T generate $SL(2,\mathbb{Z})$, $< S,T >= SL(2,\mathbb{Z})$.

For $\Omega \subset M$, any open subset of M, the usual inner product is defined by Poincaré disk model, the group of isometries is given by $SU(1,1)$, which is isomorphic to $SL(2,\mathbb{R})$. The disk model is a warped cone with boundary $\mathbb{S}^1$.

The Cayley transform takes the upper half plane to and from the Poincaré disk

$$\mathbb{R}^{2^+} \longrightarrow \mathbb{D}$$

$$(x, y) \mapsto (\frac{2x}{x^2 + (1+y)^2}, \frac{x^2 + y^2 - 1}{x^2 + (1+y)^2}). \tag{2.12}$$

If we use complex numbers, then $u \to \frac{u+i}{iu+1}$ maps the disk model to the half-plane. The boundary of $\mathbb{D}$, $\mathbb{S}^1$, gets unwrapped and stretched on $\mathbb{R}$. Take a look for an example at [8]. Poincaré disk model is good because $\mathbb{S}^1$ shows as the boundary, on the other hand half-plane has the above interesting group action and looks more like a cylinder.

If we confine ourselves to the subgroup $SL(2, \mathbb{Z})$, then the Dirac operator would be invariant as $SL(2, \mathbb{Z})$ is the discrete group of isometries.

From the geometry of the Poincaré disk one obtains the following exact sequence

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C_\nu(\mathbb{D}) \longrightarrow C(\mathbb{S}^1) \longrightarrow 0, \tag{2.13}$$

where $C_0(\mathbb{D})$ is the space of bounded continuous functions that go uniformly to zero at infinity, $C_\nu(\mathbb{D})$ the space of bounded continuous functions with uniform radial limit, and $C(\mathbb{S}^1)$ is the quotient.

Chapter 3

Cycles, Unbounded Cycles, Equivariant Cycles and Equivariant Extensions

This is an expository chapter on Kasparov cycles and equivariant cycles and the unbounded versions of these two − exact sequences and equivariant exact sequences and the Busby map of an exact sequence − Baaj−Julg machinery, and the Julg and Skandalis theorem.

KK-theory has its roots in two theories: the Brown−Douglas−Filmore theory on the extensions of commutative C^*-algebras [11], and Atiyah's axiomatization of the properties of elliptic operators on closed manifolds. The passage from unbounded cycles to Bounded, original, KK-cycles is furnished by bounded transform. This surjective map found by Baaj and Julg, from unbounded cycles to bounded classes, turns into an isomorphism by defining an equivalent relation given by a homotopy on unbounded cycles[25].

We know that homology and cohomology occur in pairs. To each homology theory there corresponds a cohomology theory, and vice versa by duality. The Atiyah-Hirzebruch cohomology theory is called K-theory[2],[3]. So it makes sense to call the corresponding homology K-homology.

KK-theory is a common generalization both of K-homology and K-theory as an additive bivariant functor on separable C^*-algebras. Using Kasparov's KK-theory [28], we have :

$$K_0^a(X) \cong KK^0(C(X), \mathbb{C}), K_1^a(X) \cong KK^1(C(X), \mathbb{C}).$$

Through out this chapter A and B are $\mathbb{Z}_2$-graded C^*-algebras. We also use the notation I_ϕ for the set $\{T \in \mathcal{L}(E) : [\phi(a), T] \in \mathcal{K}(E), \forall a \in A\}$ and J_ϕ for the set $\{T \in \mathcal{L}(E) : \phi(a)T, T\phi(a) \in \mathcal{K}(E), \forall a \in A\}$ to simplify the formulas.

Definition 3.0.1. *[30] The set of Kasparov cycles, $\mathcal{E}(A, B)$, is given by triples (E, ϕ, F) satisfying the conditions:*

i) E is a countably graded $\mathbb{Z}_2$-graded Hilbert B-module;

ii) the map ϕ is a graded $*$-homomorphism $\phi : A \longrightarrow \mathcal{L}(E)$;

iii) the self-adjoint degree one operator $F \in I_\Phi$ is such that $F^2 - 1 \in J_\phi$.

The set $\mathcal{E}(A, B)$ can be made into a semigroup under the operation of direct sum, and modulo a suitable equivalence relation, a group is obtained:

Definition 3.0.2. *[30] $KK(A,B)$ is the quotient of $\mathcal{E}(A,B)$ by the equivalence relation given by homotopy.*

Remember that if G is a locally compact group and A a C^*-algebra, an action of G on A is a homomorphism $\alpha : G \longrightarrow Aut(A)$, from G into $*$-automorphisms of A such that $\alpha : G \longrightarrow A, g \mapsto \alpha_g(a)$ is continuous. A with such a group action is called a G-*algebra*.

The continuity condition is the analogue of requiring that a unitary representation be strongly continuous. It is too strong to require that $g \mapsto \alpha_g$ be norm continuous from G to the bounded operators on A. For example if the action of a locally compact group G on $C_0(M)$ is given by the left regular representation, then the action will never be norm continuous unless G is discrete.

Definition 3.0.3. *[29][49] Let A, B be G-algebras. A countably generated Hilbert B-module E acts on A through a homomorphism $A \longrightarrow \mathcal{L}_B(E)$ will be called an $A-B$-module.*

Definition 3.0.4. *A Kasparov equivariant $A-B$-module is a triple (U,π,F) (or maybe (U,π,E,F) if we want to mention the module) in which*

i) $U : G \longrightarrow \mathcal{L}(E)$ is a unitary representation. Unitary in the sense that

$$\langle U_g\xi, U_g\eta\rangle = g.\langle \xi,\eta\rangle, \forall g \in G, \xi,\eta \in E.$$

ii) $\pi : A \longrightarrow \mathcal{L}_B(E)$ is an equivariant $*$-homomorphism. Equivariant in the *sense that*

$$\pi(g.a) = U_g \pi(a) U_{g^{-1}}$$

iii) F is a self-adjoint operator in $\mathcal{L}_B(E)$.

Moreover $U, \pi,$ and F are related as follows:

$$\pi(a)(F^2 - 1), [\pi(a), F], [U_g, F]$$

are compact operators.

We note that if F is not self-adjoint, one more condition needs to be satisfied: $\pi(a)(F^* - F) \in \mathcal{K}_B(E)$.

If the grading is even, we call the cycle even, and if it is not graded, trivially graded, we call it odd.

Definition 3.0.5. *[19][25]*

Suppose $(A, \alpha), (B, \beta)$ are G-algebras. An unbounded equivariant

$Kasparov$ $(A, B)-module$ $(\mathcal{A}, E_B, \mathcal{D})$ consists of a dense $*$-subalgebra $\mathcal{A} \subset A$, a countably graded module E_B over B, a homomorphism V from G into invertible degree zero bounded linear (not necessarily adjointable) operators on bi-module E, a graded homomorphism $\phi : A \longrightarrow \mathcal{L}_B(E)$, and an odd, self-adjoint, regular operator $\mathcal{D} : dom(\mathcal{D}) \longrightarrow E$ that satisfy the following relations

i) $V_g(\phi(a)eb) = \phi(\alpha(a))V_g(e)\beta_g(b)$, and $\langle V_g e, V_g f\rangle_B = \beta_g(\langle e, f\rangle_B), \forall g \in G, a \in \mathcal{A}, e \in E,$ and $b \in B$;

ii) $\phi(a)$ preserves dom($\mathcal{D}$), $\phi(a)(dom\mathcal{D}) \subset \mathcal{D}$, and the graded commutator $[\mathcal{D}, \phi(a)]$ is bounded(not compact),$\forall a \in \mathcal{A}$;

iii) $\phi(a)(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_B(E), \forall a \in \mathcal{A}$;

iv) V_g preserves dom($\mathcal{D}$) and commutes with $\mathcal{D}$, $[\mathcal{D}, V_g] = 0, \forall a \in \mathcal{A}$.

As equivariant cycles in KK_G are KK- cycles, we review on the non-equivariant case first.

Let $m \in \mathcal{L}_B(E), n \in \mathcal{L}_C(F)$.

Theorem 3.0.6. *[30] Let $E := E_1 \hat{\otimes}_{\phi_2} E_2$. The cycle $(E, \phi_1 \hat{\otimes} Id, F)$ in $KK(A, C)$ is a Kasparov product of two cycles (E_1, ϕ_1, F_1), and (E_2, ϕ_2, F_2) in $KK(A, B)$, and $KK(B, C)$ respectively, if*

i) The operator $T_x : E_2 \longrightarrow E_2$ defined by $T_x(e) := x \hat{\otimes} e$ satisfies

$$T_x \circ F_2 - (-1)^{\partial x} F \circ T_x \in \mathcal{K}(E_2, E); \tag{3.1}$$

ii) $[F_1 \hat{\otimes} Id, F]$ is of the form $P + K$ for some $K \in J_{\phi_1 \hat{\otimes} Id}$, and positive $P \in I_{\phi_1 \hat{\otimes} Id}$.

The first condition will be referred to as the *connection* condition, and the second condition as the *positivity* condition. It is known that a product satisfying the above conditions always exists and is unique up to the above equivalence relation.

Remark 3.0.7. *[30] The Hilbert module connection condition can be defined by restricting the domain of $x \in E_1$:*

í) $T_x : E_2 \longrightarrow E$ *defined by* $T_x(e) = x\hat{\otimes}e$ *satisfies*

$$T_x \circ F_2 - (-1)^{\partial x} F \circ T_x \in \mathcal{K}(E_2, E);$$

for every homogeneous element $x \in \phi_1(A)E_1$.

Remark 3.0.8. *In order to use non-self-adjoint regular operators we have to pose more conditions. In fact*

i) $(1 + D^2)^2$ *should be replaced by* $(1 + DD^*)^2$;

ii) $\mathfrak{Dom}D = \mathfrak{Dom}D^*$;

iii) $(1 + D)^{-1} \in J_\phi$.

Baaj and Julg introduced the concept of an *unbounded Kasparov module.* The set of unbounded modules is denoted $\Psi(A, B)$. Baaj and Julg showed the existence of a natural surjection $\beta : \Psi(A, B) \longrightarrow KK(A, B)$. Hence, every KK-element can be represented by an unbounded module. As we mentioned in the beginning of this chapter, the introduction of unbounded modules by Baaj and Julg was mainly motivated by the existence of a simple formula for the product

$$\Psi(A_1, B_1)\hat{\otimes}_{\mathbb{C}}\Psi(A_2, B_2) \longrightarrow \Psi(A_1\hat{\otimes}_{\mathbb{C}}A_2, B_1\hat{\otimes}_{\mathbb{C}}B_2). \tag{3.2}$$

3.1 Summary of the Baaj–Julg method

First of all we recall the definition of bounded transform, which is used as the main tool to get a Kasparov module from an unbounded cycle.

Definition 3.1.1. *Suppose T is a densely defined closed operator on a Hilbert space $\mathcal{H}$. It can be shown (Proposition 3.18 [42]) that $C_T := (I + T^*T)^{-1}$ is a positive bounded self-adjoint operator, so it has a positive square root $C_T^{1/2}$. The operator $Z_T := TC_T^{1/2} = T(I + T^*T)^{-1}$ is called the bounded transform of T.*

Now we want to explain these constructions. Set

$$b : \mathbb{R} \longrightarrow \mathbb{R}$$

$$b(x) = x(1 + x^2)^{-\frac{1}{2}}. \tag{3.3}$$

Using a functional calculus for bounded continuous functions of an unbounded regular self-adjoint operator in [4, 30] we can define $b(D) := D(1 + D^2)^{-\frac{1}{2}}$.

Lemma 3.1.2. *[30] Suppose D is a regular self-adjoint unbounded operator. Then the resolvent of D is in J_ϕ if and only if $(1 + D^2)^{-1}$ is in J_ϕ.*

Proof. $f(r) := \frac{(1+r^2)^{\frac{1}{2}}}{i+r}$ is bounded on $\mathbb{R}$. So, the polar decomposition

$$(i + D)^{-1} = f(D)\{(1 + D^2)^{-1}\}^{\frac{1}{2}}$$

shows that $(1+D)^{-1}\phi(a)$ is compact for all a in A. In the same way, $\phi(a)(1+D)^{-1}$ is compact. On the other hand if $(i+D)^{-1} \in J_\phi$, then $(-i+D)^{-1} \in J_\phi$ and so $(i+D)^{-1}(i+D)^{-1*} = (i+D)^{-1}(-i+D)^{-1} = (1+D^2)^{-1} \in J_\phi$ ■

Now we state some identities which will be used later.

Lemma 3.1.3. *[4, 30] If D is a regular and self-adjoint on E, then*

$$\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} D(1+\lambda+D^2)^{-1}x\, d\lambda = b(D)x.$$

Proof. For the bounded case we have ordinary functional calculus, but for the unbounded case we are using a functional calculus which was given in [30]. In fact, it suffices to estimate

$$\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} r(1+\lambda+r^2)^{-1}x\, d\lambda$$

by Riemann sums and then the above mentioned functional calculus. ■

Lemma 3.1.4. *[30, chapter 4] Let D be a self-adjoint regular unbounded operator on E, and T a bounded adjoinable operator. If $\lambda \geq 0$, and we define*

$C(x,y) := \langle Dx, Ty\rangle - (-1)^{\partial D \partial T}\langle T^* x, Dy\rangle$;

$T_0 := (1+\lambda)^{\frac{1}{2}}(1+D^2+\lambda)^{-1}$; *and*

$T_1 := D(1+\lambda+D^2)^{-1}$. *then*

$$\langle x, [T_1, T]y\rangle = C(T_0 x, T_0 y) - (-1)^{\partial T \partial D} C(T_1 x, T_1 x).$$

Proof. If we set $a := (-1)^{\partial D \partial T}$, then

$$
\begin{aligned}
(T_0x, T_0y) - (-1)^{\partial T \partial D} C(T_1x, T_1x) &= \langle \frac{D^2}{1+\lambda+D^2}x, T\frac{D}{1+\lambda+D^2}y\rangle \\
&+ (1+\lambda)\langle \frac{D}{1+\lambda+D^2}x, T\frac{1}{1+\lambda+D^2}y\rangle \\
&= \langle T^*\frac{D}{1+\lambda+D^2}x, \frac{D^2}{1+\lambda+D^2}y\rangle \\
&+ a(1+\lambda)\langle T^*\frac{1}{1+\lambda+D^2}x, \frac{D}{1+\lambda+D^2}y\rangle \\
&= a\langle x, T\frac{D}{1+\lambda+D^2}y\rangle + \langle T^*\frac{D}{1+\lambda+D^2}x, y\rangle
\end{aligned}
$$

which is the left hand side. ■

Proposition 3.1.5. *[4] Suppose $(E, D) \in \Psi(A, B)$. Let $F := D(1 + D^2)^{\frac{-1}{2}}$. Then (E, F) is a Kasparov bimodule. The above proposition defines a map $\beta : \Psi(A, B) \longrightarrow KK(A, B)$, by $\beta(E, D) := (E, D(1 + D^2)^{\frac{-1}{2}})$.*

Proposition 3.1.6. *[4] $\beta : \Psi(A, B) \longrightarrow KK(A, B)$ is surjective.*

Lemma 3.1.7. *Suppose E is a Hilbert B-module, and D a self-adjoint regular operator on E. If $(1 + D^2)^{-1} \in J_\phi$, and the domain of D is invariant under operators T, T^*, and both $[D, T], [D, T^*]$ are bounded, then $[b(D), G] \in J_\phi$.*

3.1.1 A short summary on extensions of C^*-algebras

As KK-theory has one of its roots in the theory of extensions of C^*-algebras, here we present a summary related to this topic. The content of this section is mainly taken from [9], [10], [50], and [24].

Definition 3.1.8. *[50] Let A and C be two C^*-algebras. An **extension** $E = (\alpha, B, \beta)$ of A by C is an exact sequence of C^*-algebras and morphisms*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

If the projection map, β, has a section, then the extension is called trivial. This means that there exists a morphism $\gamma : C \longrightarrow B$ with $\beta \circ \gamma = Id_C$.

So an extension $E = (\alpha, B, \beta)$ tells us the way A sits in B and the isomorphism $\frac{B}{A} \cong C$ tells us about the size of A relative to B.

Many authors prefer reversed terminology and say that B is an extension of C by A. This reversed definition, even though somewhat unnatural, will be our definition of an extension because it goes well with the traditional notation of KK-theory and because of sources we are using.

In fact we are interested in extensions because $KK^1(A, B)$ has a realization as classes of extensions [24, chapter 3]. In [44] it is shown that such a description exists for the equivariant case too.

All of the information in an extension

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

can be concentrated in a map, called the Busby invariant of the extension. The Busby map of the above exact sequence is explicitly given as follows:

Definition 3.1.9. *[44] Let $\mathcal{M}(A)$ denote the multiplier algebra of the C^*-algebra A as in 2.0.9 and $\pi : B \longrightarrow \mathcal{M}(A)$ be the $*$-homomorphism determined by the extension, i.e. $\pi(b)a := \alpha^{-1}(b\alpha(a))$.*

Now suppose $c \in C$ is given and find a b in the pre-image of c under β.

Definition 3.1.10. *[50, lemma 3.2.4] The* ***Busby invariant*** *$\tau_E : C \longrightarrow \frac{\mathcal{M}(A)}{A}$ is defined by $\tau_E(c) := q_A(\pi(b))$, in which $q_A : \mathcal{M}(A) \longrightarrow \frac{\mathcal{M}(A)}{A}$ is the quotient map.*

As noticed in 2.0.5 there exists an exact sequence

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C_\nu(\mathbb{D}) \longrightarrow C(\mathbb{S}^1) \longrightarrow 0. \tag{3.4}$$

The Busby map for this exact sequence is given by

$$\tau : C(\mathbb{S}^1) \longrightarrow \frac{\mathcal{M}(C_0(\mathbb{D}))}{C_0(\mathbb{D})} = \frac{C_b(\mathbb{D})}{C_0(\mathbb{D})}, f \mapsto \tilde{f} + C_0(\mathbb{D}) \tag{3.5}$$

in which $\tilde{f}$ is an extension of f from the boundary to the disk.

We are reminded here that a C^*-dynamical system (A, G, α) is a homomorphism $\alpha : G \longrightarrow Aut(A)$ form a group G into $*$-automorphisms of A such that for every $a \in A$, $g \mapsto \alpha_g(a)$ is continuous from G to A with the C^*-norm.

Definition 3.1.11. *A G-algebra A is a C^*-algebra A, which is acted on by a group G and the action is continuous in the above sense.*

Equivariant extensions are defined naturally as extensions of G-algebras in which the morphisms are equivariant.

We say that two G-extensions

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0, \ \ and$$

$$0 \longrightarrow B \longrightarrow F \xrightarrow{q} A \longrightarrow 0$$

are unitarily equivalent if there is a unitary $u \in \mathcal{M}(B)$ such that $g.u - u \in B$ for all $g \in G$, and the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & Adu\big\downarrow & & \big\downarrow & & || & & \\ 0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We note that in this definition Adu and the middle vertical map do not need to be equivariant. The sum of two equivariant extensions is defined in the same way as the non-equivarianve case. So the sum of the above two G-extensions is identical to

$$0 \longrightarrow M_2(B) \longrightarrow X \longrightarrow A \longrightarrow 0$$

in which $X := \{ \begin{pmatrix} e & b_1 \\ b_2 & f \end{pmatrix} ; b_1, b_2 \in B, e \in E, f \in F, p(e) = q(f)\}$, and the action of G on X is given component-wise: $g. \begin{pmatrix} e & b_1 \\ b_2 & f \end{pmatrix} := \begin{pmatrix} g.e & g.b_1 \\ g.b_2 & g.f \end{pmatrix}$.

In the rest of this chapter we give a summary of a series of papers [45],[46],[44] by Klaus Thomsen that we use in our work. First we recall that all information about an extension

$$0 \longrightarrow B \xrightarrow{q} E \xrightarrow{p} A \longrightarrow 0$$

can be summed up in its Busby map $\tau; B \longrightarrow Q(A)$. The idea is the same for equivariant extensions as working with homomorphisms is in general easier. It is almost straight forward to show that to every G-equivariant extension

$$0 \longrightarrow (B, \beta) \xrightarrow{q} (E, \gamma) \xrightarrow{p} (A, \alpha) \longrightarrow 0$$

there corresponds an equivariant map $\tau \in Hom_G(A, \frac{\mathcal{M}(B)}{B})$.

For the reverse direction, that every G-equivariant map $\tau : A \longrightarrow \frac{\mathcal{M}(B)}{B}$ comes from an G-equivariant extension, one needs a technical theorem, which is proved in [44, theorem 2.1] : If α, and β are (strongly) continuous, then γ is continuous.

In the non-equivariant case it is known that there is a realization of $KK^1(A, B)$ in term of classes of invertible extensions. There is a similar result for non-equivariant case as for the equivariant case. The procedure is almost the same except that in the non-equivariant case $0 \longrightarrow B \xrightarrow{q} E \xrightarrow{p} A \longrightarrow 0$, one needs to stabilize B to get group structure on invertible extensions. In the equivariant case one needs to stabilize all algebras and this stabilization is not tensoring by $\mathcal{K}$, the compact operators on a separable

Hilbert space. Algebras should be stabilized in a way that the resulting algebras contain information about the group acting. A stabilized algebra is obtained as follows: Represent G on $\oplus_{i=1}^{\infty} L^2(G)$ by direct sum of regular representations of G on $L^2(G)$. As a result G is represented by compact operators on $L^2(G)$, which is denoted by $\mathcal{K}_G$.

Definition 3.1.12. *[24] A $*$-homomorphism $\phi : A \longrightarrow B$ is called quasi-unital when there is a projection $p \in \mathcal{M}(B)$ such that $\overline{\phi(A)B} = pB$.*

We note that every quasi-unital $*$-homomorphism $\phi : A \longrightarrow B$ has a unique extension $\underline{\phi} : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ [24, Corollary 1.1.15]. So from an action $\beta : G \longrightarrow Aut(B)$ we get endomorphisms $\beta_g : B \longrightarrow B$, its extension $\underline{\beta_g} : \mathcal{M}(B) \longrightarrow \mathcal{M}(B)$.

As we explained earlier, there is a bijection between equivariant extensions of A by B and $Hom_G(A, \frac{\mathcal{M}(B)}{B})$. We say that two extensions $\phi, \psi \in Hom_G(A, \frac{\mathcal{M}(B)}{B})$, are unitarily equivalent if there exist a unitary $u \in \mathcal{M}(B)$ such that:

i) $\widehat{\beta}_g(q_B(u)) = q_B(u)$, u is invariant;

ii) $Adq_B(u) \circ \phi = q_B(u)\phi(.)q_B(u)^* = \psi(.)$.

We say that a G-extension $\phi \in Hom_G(A, \frac{\mathcal{M}(B)}{B})$ is *degenerate* if there is a $*$-homomorphism $\overline{\phi} : B \longrightarrow \mathcal{M}(A)$ such that $q_B \circ \overline{\phi} = \phi$. We say that a G-extension ϕ is invertible if there exists another G-extension ψ such that their sum is degenerate. Now suppose that B is weakly stabilized, which means $B \otimes \mathcal{K}$ is G-isomorphic to B. Let Ext_G^h be classes of invertible G-extensions identified by homotopy. So two G-extensions ϕ, and ψ are the same if and only if there exists an invertible G-extension $\Psi \in Hom_G(A, Q(IB))$ such that $\widehat{\pi}_0 \circ \Psi = \psi$, $\widehat{\pi}_1 \circ \Psi = \phi$ in which $\pi_i : IB = C([0,1]) \otimes B \longrightarrow B$ is given by $\pi(f \otimes b) = f(i)b$ and the action on IB is given by $id \otimes \beta$. We note that $\pi_i : IB = C([0,1]) \otimes B \longrightarrow B$ is quasi-unital so the extension $\underline{\pi_i} : \mathcal{M}(IB) = \mathcal{M}(C([0,1]) \otimes B) \longrightarrow \mathcal{M}(B)$ exists. For the existence of $\widehat{\pi}_i$ look at [12, theorem 4.3]. Let Ext^h_G denote the homotopy classes of invertible extensions.

Theorem 3.1.13. *[44] A G-extension*

$$0 \longrightarrow (B \otimes \mathcal{K}_G, \beta \otimes \rho) \longrightarrow (E, \gamma) \xrightarrow{p} (A \otimes \mathcal{K}_G, \alpha \otimes \rho) \longrightarrow 0$$

is invertible if and only if the map

$$(E \otimes \mathcal{K}_G, \gamma \otimes \rho) \xrightarrow{p \otimes id_{\mathcal{K}_G}} (A \otimes \mathcal{K}_G \otimes \mathcal{K}_G, \alpha \otimes \rho \otimes \rho)$$

admits a completely positive equivariant contraction as a right inverse.

Definition 3.1.14. *[47] A C^*-algebra A has the lifting property if it has the following property: For every C^*-algebra B and an ideal $J \subset B$, and every completely positive contractive map $A \longrightarrow B/J$, there is a completely positive contractive lift $A \longrightarrow B$.*

Remember also

Definition 3.1.15. *[43] A G-extension* $0 \longrightarrow J \xrightarrow{j} B \xrightarrow{p} A \longrightarrow 0$ *is said to be equivariantly semi-split if there is a completely positive contraction* $s : A \longrightarrow B$ *such that* $p \circ s = id_A$ *and there is a G-equivariant countable approximate unit for j.*

Remember that a G-equivariant approximate unit for an equivariant map $j : (A, \alpha) \longrightarrow (B, \beta)$ *is a sequence* $\{u_n\} \subset \{m \in \mathcal{M}(B) : mB \subset j(A), 0 \leq m \leq 1\}$ *such that* $\bar{\beta}(u_n) = u_n$ *and* $\lim_{n\to\infty} u_n j(a) = j(a), \forall a \in A$ *and* $g \in G$.

Baaj and Skandalis showed that to have the expected behaviour in KK-theory, a G-extension should be equivariantly semi-split. They also showed that from every completely positive graded semi-spliting map one can construct a $\mathbb{Z}$-graded positive equivariant section as follows:

Proposition 3.1.16. *[5, proposition 7.16] Let* μ *be a Haar averaging on* (G, δ) *and* $0 \longrightarrow J \longrightarrow A \xrightarrow{\beta} A/J \longrightarrow 0$ *an G-equivariant exact sequence of* $\mathbb{Z}_2$*-graded* C^**-algebras which is semi-split. Suppose also that co-actions of the* C^**-algebras in the above sequence are injective. Then each completely positive graded semi-splitting* σ *of* β*, provides a* $\mathbb{Z}_2$*-graded positive equivariant section for* β.[1]

Thus we obtain an equivariant semi-splitting.

[1]For the definition of Haar averaging, the reader is recommended to see [5, 7.8.3]. In this thesis only existence of the equivariant semi-splitting is used.

Chapter 4

The Calculation of the Index of Equivariant Dirac-Schrödinger Operators on Hyperbolic Manifolds

In this chapter, through the calculation of a couple of equivariant Kasparov products and using homotopy, we calculate the index of an equivariant Dirac-Schrödinger operator on hyperbolic manifolds. The main results of this chapter are: the introduction of $KK_G^{asymp}(A, B)$ for a pair of C^*-algebras using pulling back of the equivalence relations on the group of extensions − lemma 4.1.15, which stipulates the conditions under which the potential induces a cycle in $KK_G^{asymp}(\mathbb{C}, C_0(M)\hat{\otimes}C_1)$ − the calculation of the first Kasparov product in proposition 4.1.18 − the calculation of the second Kasparov product in proposition 4.1.20 − the third Kasparov product and homotopy of cycles in proposition 4.1.24, and finally the main theorem of the chapter, theorem 4.1.25.

As it was mentioned above, the goal of this chapter is to calculate the index over a category of manifolds, hyperbolic manifolds. It is not difficult to show that when one multiplies the Riemannian metric by a positive constant c, the sectional curvature is multiplied by $\frac{1}{c}$. Therefore, up to similarity, we can suppose that manifolds with constant sectional curvature have curvatures $-1, 0$, or 1. In fact, by the following theorem, the category of hyperbolic manifolds with big enough group actions covers all manifolds with constant negative sectional curvature.

Proposition 4.0.1. *[14, chapter 8] Let M be a complete Riemannian manifold with constant sectional curvature K $(1, 0, -1)$. Then M is isometric to $\frac{\hat{M}}{G}$, where $\hat{M}$ is S^n if $K = 1$, $\mathbb{R}^n$ if $K = 0$ or $\mathbb{H}^n$ if $K = -1$, and G is a subgroup of the group of isometries of $\hat{M}$ which acts in a totally discontinuous manner on $\hat{M}$, and the metric on $\frac{\hat{M}}{G}$ is induced from the covering $\pi : \hat{M} \longrightarrow \frac{\hat{M}}{G}$.*

Throughout this chapter M is $\mathbb{H}^n$, hyperbolic space of dimension n, in the ball model, $[0, \infty) \times \mathbb{S}^{n-1}$ with metric $dr^2 + sinh^2(r)d_{\mathbb{S}^{n-1}}$.

Definition 4.0.2. *Let $C_\nu(M)$ denote the algebra of bounded and continuous functions on M such that their radial limit with respect to r coordinate exists uniformly.*

It can be shown that $C_\nu(M)$ is a C^*-algebra. From the structure of the manifold M we get the exact sequence

$$0 \longrightarrow C_0(M) \longrightarrow C_\nu(M) \longrightarrow C(N) \longrightarrow 0$$

in which the second map is given by inclusion and the third one is given by taking a uniform limit in the r direction and N is $\mathbb{S}^n$ in the ball model $[0, \infty) \times \mathbb{S}^n$ with the singular metric $dr^2 + sinh^2(r)d_{\mathbb{S}^n}$[13]. It can be shown that the above sequence is equivariant. On the other hand if the acting group is amenable, one can build an equivariant spliting out of a given splitting using averaging process in proposition 3.1.16.

4.0.1 Compactifications $\overline{M}$ with $C(\overline{M}) = C_\nu(M)$

Through the following sections, we establish some of the properties of these compactifications. Roughly speaking these are compactifications intermediate between the one-point compactification and the non-separable Stone-Čech compactification.

First solution

In the following, let the abelian C^*-algebra $C_\nu(M)$ be the set of functions whose radial limit exists. Then we have the following exact sequence

$0 \longrightarrow C_0(M) \longrightarrow C_\nu(M) \longrightarrow C(N) \longrightarrow 0.$

All algebras above are commutative and we know that every commutative C^*-algebra is isomorphic to some $C_0(\Omega)$, where Ω is the Gelfand spectrum. In the case that the original C^*-algebra is unital, the Gelfand spectrum is compact, and so the C^*-algebra is isomorphic to $C(\Omega)$.

$C_\nu(M)$ is a unital C^*-algebra. Let $\overline{M}_1 := \Omega(C_\nu(M))$. By the Gelfand representation theorem, $C_\nu(M) \cong C(\overline{M}_1)$, where Ω is the Gelfand space of $C_\nu(M)$.

This gives one solution for a topological manifold $\overline{M}$ satisfying $C(\overline{M}) = C_\nu(M)$, and we will find two more solutions.

Another solution can be obtained in the following way.

Second solution

Proposition 4.0.3. *Let $W \subset M$ be the collar of the warped cone over (N, g_N) and let $k : (W, g_M|_W) \longrightarrow (0, \infty) \times N$ be an isometry. Finally, let $\overline{M}_2 : M \coprod N$ with the weakest topology, such that $\tilde{k} : W \coprod N \longrightarrow (0, \pi/2] \times N$ defined by*

$$\tilde{k}|W := (\arctan \times id_N) \circ k, \quad \tilde{k}|_N := (x \mapsto (\pi/2, x)),$$

is a homeomorphism.

Let $f \in C(\overline{M}_2)$, the restriction $f|_M$ defines an element of $C_\nu(M)$, such that $\lim_{r\to\infty f=f|_N}$. The restriction map $\rho : C(\overline{M}_2) \longrightarrow C_\nu(M)$, given by $f \mapsto f|_M$ is a $*$-isomorphism.

Proof. The topology defined on $\tilde{M}$ by $\tilde{k}$ induces the same topology as the original topology on M. As $\overline{M}_2$ is compact, it is complete. Now suppose $f \in C(\overline{M}_2)$, then $\lim_{r\to\infty} f(\tilde{k}^{-1}(\arctan(r), n)) = f(\lim_{r\to\infty} \tilde{k}^{-1}(\arctan(r), n)) = f(\tilde{k}^{-1}(\pi/2, n))$. Hence the radial limit of f on M exists. So, ρ is well defined.

Hence the image of a function under the map $\rho : C(\overline{M}_2) \longrightarrow C_\nu(M)$, given by restriction, is bounded and continuous.. For surjectivity, suppose $f \in C_\nu$ is given. Let $\bar{f}$ be the function on $\overline{M}_2$ extending f by adding its radial limit. Then $\rho(\bar{f}) = f$. For injectivity, if $\rho(f_1) = \rho(f_2)$, then $f_1 \circ \tilde{k}^{-1}(0, \pi/2) \times N = f_2 \circ \tilde{k}^{-1}(0, \pi/2) \times N$. Because f_1, f_2 are in $C(\overline{M}_2)$ they should agree on $f \circ \tilde{k}^{-1}(0, \pi/2] \times N$. So, $f_1 = f_2$. ■

Corollary 4.0.4. *From the above $*$-isomorphism ρ, we have $C_\nu(\overline{M}) \cong C(\overline{M}_2)$. So $\overline{M}_2$ satisfies $C(\overline{M}) = C_\nu(M)$, and as both $\overline{M}_1$ and $\overline{M}_2$ are compact spaces satisfying the equation, we have $\overline{M}_1 \cong_{homeo} \overline{M}_2$.*

Proposition 4.0.5. *The group G acts by homeomorphism on $\overline{M}_2$.*

Proof. Using the above $*$-isomorphism, we show that the warped cone (M, g_M) is asymptotically G-equivariant if and only if the G-action on M extends to a G-action on $\overline{M}$ by homeomorphism, such that the inclusion $N \hookrightarrow \overline{M}_2$ is G-equivariant.

Suppose $\rho(\bar{f} = f)$; then $f = \bar{f} \circ k^{-1}|_{(0,\pi/2)\times N}$. For points in $(0, \pi/2) \times N$ define the action $g.\bar{f} := f \circ g^{-1}$, and for $(0, \pi] \times N$ let $g.\bar{f} = (\lim f) \circ g^{-1}$.

We note that if a group of isometries acts on M, then by functoriality, it acts on any function space on M. So a group of isometries acting on M induces an action of the same group on $C(\overline{M}_2)$. ■

Third solution, geometric solution

The third way of solving the equation $C(\overline{M}) = C_\nu(M)$ is explained through the following paragraphs. We are going to put a geometric structure on $\overline{M}$ and turn it into a manifold with boundary. Clearly $\overline{M}_{geo}$ is a compactification of M.

Remember that the limit of a sequence of quadratic forms is a quadratic form.[1] From the Riemnannian metric on M we construct a possibly singular Riemannian metric on $\overline{M}$. Let $\overline{M}_{geo}$ be the manifold with boundary obtained from $M \bigsqcup N$ by perturbing the warping function to bound the Riemannian metric on M and defining it on N; By polarization we get a symmetric bilinear form. We state this formally:

Definition 4.0.6. *Suppose we have perturbed the warped function so that the metric is bounded at infinity. Let $\overline{M}_{geo}$ denote the Riemannan manifold obtained from $\overline{M}$ with the metric given by*

$$\langle , \rangle_x := \lim_{n\to\infty} \langle , \rangle_{x_n}, x \in \overline{M}, \forall\ x_n \mapsto x.$$

We can thus identify the geometrical and topological models of $\overline{M}$. From now on, $\overline{M}$ will stand for either the topological $\overline{M}$ or its geometric version. The continuous functions on $\overline{M}$ coincide on M with the functions in $C_\nu(M)$, having radial limit. Thus $C(\overline{M}) \cong C_\nu(M)$, and therefore we can as well identify $\overline{M}$ and $\overline{M}_{geo}$.

A state of a C^*-algebra is a positive linear functional of norm one. When the C^*-algebra is unital this is equivalent to being a unital positive functional.

Definition 4.0.7. *A state ϕ on a unital C^*-algebra is said to be faithful if it*

[1] [22, problem 1 chapter one]

takes positive and non-zero elements to positive and non-zero elements.

Before we start proving the following proposition, we recall the known result that:

Theorem 4.0.8. *(**Banach–Alaoglu**) [22] The state space of a unital C^*-algebra is weak* closed and weak* compact.*

In the proof of the next following proposition we need a result which for the convenience of the reader we mention here:

Theorem 4.0.9. *(**Riesz–Markov–Kakutani**) [22] Let X be a locally compact Hausdorff space. For any positive linear functional ψ on $C_c(X)$, there is a unique regular Borel measure μ on X such that*

$$\forall f \in C_c(X),\ \psi(f) = \int_X f(x) d\mu(x).$$

Proposition 4.0.10. *$C(\overline{M})$ is a separable C^*-algebra and has a faithful state.*

Proof. Remember that every positive faithful linear functional on a separable space of functions $C(X)$ induces a measure on X. Now as a manifold is second countable, it is separable and hence the function algebra $C_0(M)$ is separable. For the same reason $C(N)$ is separable and from the above exact sequence we deduce that $C(\overline{M})$ is separable.

Since $C_0(M)$ and $C(N)$ have a countable dense subset we can find a dense subset of $C_\nu(M)$ in

$$0 \longrightarrow C_0(M) \longrightarrow C_\nu(M) \longrightarrow C(N) \longrightarrow 0.$$

The existence of a countable dense subset guarantees the existence of a faithful linear functional. In fact pick a dense sequence y_n in $C(\overline{M})_+$ and by the Hahn−Banach theorem let ϕ_{y_n} be positive and nonzero at y_n, then $\sum 2^{-n}\phi_n$ is a positive linear functional on $C(\overline{M})$. ■

4.1 Amenability

Amenability is related to the Banach-Tarski paradox, and provides good behaviour to exotic spaces associated with amenable groups. Here we give a definition. Of course the definition can be formulated in different ways. We pick a definition using which it is easier to check amenability of a group. First recall that if G is a locally compact group, then $\ell^\infty(G)$ is the set of bounded functions on G equipped with the supremum norm.

Definition 4.1.1. *[17] Let G be a locally compact group and X be a closed subspace of $\ell^\infty(G)$ containing the constant functions and closed under complex conjugation. A linear functional $m : X \longrightarrow \mathbb{C}$ is called a mean on X if*

i) $m(\bar{f}) = \overline{m(f)}$;

ii) $m(1) = 1$; *and*

iii) $f \geq 0 \implies m(f) \geq 0$.

A mean is left invariant if $m(\lambda_s(f)) = m(f), \forall s \in G, \forall f \in X$, *where* $\lambda_s(f)(x) := f(s^{-1}x)$.

A discrete group G *is called amenable if there is a left-invariant mean on* $\ell^\infty(G)$.

We use the following theorem when applying amenability.

Theorem 4.1.2. *[17] A locally compact group* G *is amenable if and only if any continuous affine action of* G *on a non-empty subset of a locally convex topological vector space has a fixed point.*

Lemma 4.1.3. *There exists an invariant measure* μ_ϕ *on* $\overline{M}$. *Denote by* $L^2(\overline{M}, \phi)$ *, where* ϕ *is the state associated with the invariant measure, the space of classes of functions given by this measure.*

Proof. The amenable group G acts on the state space of $C(\overline{M})$. Since $C(\overline{M})$ is unital, the state space of a unital C^*- algebra is weak* compact. So there exists an invariant state ϕ. Now from the action of the group on this compact space we get an invariant mean. By the Riesz−Markov−Kakutani-theorem this mean corresponds to a measure on $\overline{M}$. Let $L^2(\overline{M})$ be the space of L^2-sections on $\overline{M}$ given by this measure. ■

Definition 4.1.4. *[44, section 3.2] A semi-spliting for an extension* $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ *is a completely positive contraction* $A \longrightarrow E$ *which is right inverse for* $E \longrightarrow A$.

Assume that the exact sequence $0 \longrightarrow C_0(M) \longrightarrow C_\nu(M) \longrightarrow C(N) \longrightarrow 0$ has a semi-splitting map that we denote by s and it is a positive linear map $C(N) \longrightarrow C_\nu(M)$.

Lemma 4.1.5. *$L^2(N, \phi \circ s)$ is naturally isomorphic to a subspace of $L^2(M, \phi)$.*

Proof. Let $\phi \circ s$ be the composition of the above state ϕ and the semi-spliting s. Out of this positive linear map $\phi \circ s$, one can build a measure using the Riesz−Markov−Kakutani theorem Let $L^2(N, \phi \circ s)$ denotes the equivalence class of the functions and the inner product given by integration. Let $C(M, S)$ denote the space continuous function from M to an inner product space S. One can use the above invariant state to find the completion of $C(M, S)$. The corresponding space is shown by $L^2(M, S)$ and is called the space of L^2- functions with coefficients in S. The $L^2(N, \phi \circ s)$ space constructed this way is obtained by applying ϕ to $s(C(N))$. Note that if $s(f) = 0$ for some $f \in C(N)$ the $p \circ s(f) = f = 0$, where p is the projection map in the above exact sequence. ■

Let E be the Dirac bundle on $\overline{M}$ and $i : N \longrightarrow \overline{M}$ be the inclusion map. Let $i^*(E)$ be Dirac bundle on N pulled back by inclusion. As $\overline{M}$ and N are both compact, by Swan's theorem we can find a non-negative n such that $E \subset \mathbb{M}_n \otimes C(\overline{M}) =: M_n(\overline{M})$ and $i^*(E) \subset \mathbb{M}_n \otimes C(N) =: M_n(N)$.

As tensoring by the matrix algebra $\mathbb{M}_n$ is an exact functor, by tensoring $0 \longrightarrow C_0(M) \longrightarrow C(\overline{M}) \underset{s}{\overset{p}{\rightleftharpoons}} C(N) \longrightarrow 0$ with $\mathbb{M}_n$, we get the following exact sequence

$$0 \longrightarrow M_n(M) \longrightarrow M_n(\overline{M}) \underset{s_*}{\overset{p_*}{\rightleftharpoons}} M_n(N) \longrightarrow 0$$

As $p : C(\overline{M}) \longrightarrow C(N)$ is a homomorphism, by the Gelfand dualization, we have a map $i : N \longrightarrow \overline{M}$. Using this map one can pull back the spinor bundle E on $\overline{M}$ to a bundle $i^*(E)$ over N.

Definition 4.1.6. *We define the Dirac operator on N using this bundle and the corresponding L^2-space $L^2(N, i^*E)$. So $D_N := D_M|_{s(L^2(N,i^*(E)))}$.*

In this way one can go back and forth between L^2-spaces using

$$L^2(\overline{M}, E) \underset{s_*}{\overset{p_*}{\rightleftharpoons}} L^2(N, i^*E) \longrightarrow 0. \tag{4.1}$$

So

$$D_N := D_M|_{s(L^2(N,i^*(E)))}. \tag{4.2}$$

4.1.1 $KK_G^{asymp}(A, B)$

In this section all algebras $A, A_1, A_2, ...$ are separable and algebras $B, B_1, B_2, ...$ are stable. The goal of this expository sub-section is to give a summary on the isomorphism of $KK^1(A, B)$ and $Ext^{-1}(A, B)$. Remember that $Ext^{-1}(A, B)$, the set of invertible extensions, is defined to be the set of extensions

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0 \tag{4.3}$$

in which the quotient map admits a completely positive linear contraction section.

Also we remind that $KK^1(A,B) := KK(A,B_1)$, where $B_1 = B \oplus B$ with odd grading. First we give a more obvious characterization of the cycles of $KK^1(A,B)$.

Definition 4.1.7. *[28, section 5,6] Let $\epsilon(A,B)$ consist of cycles (F,ϕ) where $F \in \mathcal{M}(B)$ and $\phi \in Hom(A, \mathcal{M}(B))$ satisfying*

i) $F\phi(a) - \phi(a)F \in B$,

ii) $(F^* - \nu)\phi(a) \in B$,

iii) $(F^2 - \nu)\phi(a) \in B$,

for all a in A.

Definition 4.1.8. *A cycle (F,ϕ) is called degenerate if $F\phi(a) - \phi(a)F = (F^* - F)\phi(a) = (F^2 - F)\phi(a) = 0$.*

Let $\delta(A,B)$ denote the set of degenerate cycles.

Definition 4.1.9. *Two pairs (F_1,ϕ_1) and (F_2,ϕ_2) are unitarily equivalent if there exist a unitary element $u \in \mathcal{M}(B)$ such that $\phi_2(a) = u\phi_1(a)u^*$ and $F_2 = uF_1(a)u^*$.*

Definition 4.1.10. *Two pairs (F_1,ϕ_1) and (F_2,ϕ_2) are called homological if $F_1\phi_1(a) - F_2\phi_2(a) \in B$.*

Let $\overline{\epsilon}(A,B)$ be the the set of above cycles modulo unitary equivalence and homology, and $\overline{\delta}(A,B)$ be the image of degenerate cycles under the above relations. Set $Ext^1(A,B) := \overline{\epsilon}(A,B)/\overline{\delta}(A,B)$.

Lemma 4.1.11. *[28, section 5,6]*

$$KK^1(A,B) \longrightarrow Ext^1(A,B)$$

$$(\eta, \phi : A \longrightarrow \mathcal{M}(\mathcal{K}_{(B)}\hat{\otimes}C_1), F) \mapsto \nu.\phi \text{ } mod \text{ } \mathcal{K}(B) \tag{4.4}$$

defined at the level of cycles induces an isomorphism.

Here we are using the of isomorphism $Ext^1(A,B)$ and the group of Busby maps.

We change the above isomorphism a little by setting $(F,\phi) \mapsto \pi(F\phi F)$. We notice this map remains surjective. This can be shown using relations $F^2 - F \in \mathcal{K}(B)$ and $[F,\phi] \in \mathcal{K}(B)$ and the fact that every cycle is equivalent to a cycle with a self-adjoint operator.

Definition 4.1.12. *Suppose A and B are two G-C^*-algebras. We define $KK_G^{asymp}((A,G),(B,G))$ by pulling back the homotopy equivalence on $Ext(A,B)$. More precisely, two cycles in $KK^1(A_G,B_G)$ are equivalent if their images under the above isomorphism are homotopic,*

$$KK_G^{asymp}(A,B) := \frac{KK^1((A,G),(B,G))}{\sim}. \tag{4.5}$$

In the same way the Baaj−Julg version from asymptotically equivariant unbounded cycles to asymptotically equivariant bounded cycles is defined. So if we set $\Psi^U((A,G),(B,G))$ to be the set of unbounded cycles and $\Psi((A,G),(B,G))$ to be the set of bounded asymptotically equivariant cycles, then there is a map b from $\Psi^U((A,G),(B,G))$ to $\Psi((A,G),(B,G))$ given by the bounded transform.

As we will see, $[r] := \big[\big(C_0(M)\otimes C_1, e : C(N) \longrightarrow \mathcal{L}_{C_0(M)\oplus C_0(M)}(C_0(M)\otimes C_1), r\big)\big]$, where e is given by multiplication by extension map and r is multiplication by the radial component, is an asymptotically equivariant unbounded cycle. Under the above isomorphism $\big(C_0(M)\otimes C_1, e : C(N) \longrightarrow \mathcal{L}_{C_0(M)\oplus C_0(M)}(C_0(M)\otimes C_1), r\big)$ is mapped to $\pi(b(r)eb(r))$.

Proposition 4.1.13. *The Busby map associated with the cycle $[r]$ equals the Busby map associated to the exact sequence*

$$0 \longrightarrow C_0(M) \longrightarrow C_\nu(M) \longrightarrow C(N) \longrightarrow 0.$$

Proof. Based on the definition of cycle $[r]$ we have

$$\begin{aligned}\pi((b(r)eb(r)) &= \pi((b(r)b(r)e) \\ &= \pi(\frac{r^2}{1+r^2}e(f)) = [e(f)] + C_b(M).\end{aligned}$$

It can be shown that the Busby map, $b : C(N) \longrightarrow \frac{C_b(M)}{C_0(M)}$, for the exact sequence $0 \longrightarrow C_0(M) \longrightarrow C_\nu(M) \longrightarrow C(N) \longrightarrow 0$ is given by $b(f) = \tilde{f}\ mod\ C_0(M)$, where $\tilde{f}$ is any function that has f as radial limit at infinity. So from the construction of $e(f)$ we see that $b(f) = \pi((b(r)eb(r)) = \pi((b(r)b(r)e)$. ∎

For each $f \in C(N)$, let $e(f)$ be the multiplication operator given by restriction to M of the extension of f along the lines perpendicular to the boundary.

Definition 4.1.14. *We call an operator $A : C(M,S) \longrightarrow C(M,S)$ asymptotically equivariant if it is equivariant at infinity; more precisely, if $Ag - A$ is bounded at infinity.*

Lemma 4.1.15. *Let A be equivariant at infinity and iA be a skew adjoint operator and A^2 becomes arbitrary large outside some compact set of M. Let $\Gamma_0(E) \oplus \Gamma_0(E) \oplus$ be odd graded and $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ be an odd operator on $\Gamma_0(E) \oplus \Gamma_0(E)$. The triple is defined by*

$$\big(C_0(M)\hat{\otimes}C_1, e \otimes 1 : C(N)\hat{\otimes}C_1 \longrightarrow \mathcal{L}_{C_0(E)\hat{\otimes}C_1}(\Gamma_0(E)\hat{\otimes}C_1), \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}) \tag{4.6}$$

in which e is the map given by multiplication given by the extended function, defines a cycle in $KK_G^{asymp}(C(N)\hat{\otimes}C_1, C_0(M)\hat{\otimes}C_1)$.

Specially, Let V be an operator on $\Gamma_0(E)$, then

$$\big[\big(C_0(M)\hat{\otimes}C_1, 1 : C_1 \longrightarrow \mathcal{L}_{C_0(M)\hat{\otimes}C_1}(C_0(M)\hat{\otimes}C_1), \begin{pmatrix} V & 0 \\ 0 & -V \end{pmatrix})\big]$$

defines a cycle in

$$KK_G^{asymp}(\mathbb{C}, C_0(M)\hat{\otimes}C_1).$$

We need the following special cases of the above lemma:

Lemma 4.1.16. *When in the special case r is the multiplication operator we get a cycle*

$$\big[\big(C_0(M)\hat{\otimes}C_1, e : C(N)\otimes C_1 :\longrightarrow \mathcal{L}_{C_0(M)\hat{\otimes}C_1}(C_0(M)\hat{\otimes}C_1), \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\big)\big].$$

Proof. The fact that the second triple defines an (asymptotically) equivariant cycle is standard; a calculation is given in Appendix A.0.2. So we check the condition of compact resolvent and equivariance for the first cycle. For compactness of resolvent, we notice that the resolvent $(i+r)^{-1} \in C_0(M)$ and $\mathcal{K}_{C_0(M)}(C_0(M)) = C_0(M)$.

We will show that $r \circ g - r$, where g is an isometry form the group acting on the manifold, is bounded at infinity. Suppose (r_0, n_0) and (r_1, n_1) are two points on the manifold; remember that $r(r_i, n_i) = r_i$. Remember also that the metric on the manifold is given by $dr^2 + f(r)ds^2$. We know that the corresponding metric distance, let's denote it by d, is given by infimum over integral of paths joining two points. So

$$|r_0 - r_1| \leq d((r_0, n_0), (r_1, n_1)).$$

Let $\epsilon = d((r_0, n_0), g(r_0, n_0))$. As g is an isometry and using triangle inequality we have

$$
\begin{aligned}
d((r_0, n_0), (r_1, n_1)) &= d(g(r_0, n_0), g(r_1, n_1)) \\
&\leq g((r_0, n_0), g(r_1, n_1)) + \epsilon.
\end{aligned}
$$

Subtracting ϵ from the sides of this equation and using the previous inequality we have

$$
\begin{aligned}
|r_0 - r_1| - \epsilon &\leq d((r_0, n_0), (r_1, n_1)) - \epsilon \\
&\leq d((r_0, n_0), g(r_1, n_1)).
\end{aligned}
$$

Now fix r_1 and allow n_1 change in $\mathbb{S}^{n-1}$. As g is an isometry, $g|_{\mathbb{S}^{n-1}}$ is an isometry of $\mathbb{S}^{n-1}$.

So $g(r_1, n) = (r_1, n_0)$ for some $n \in \mathbb{S}^{n-1}$. So $d((r_0, n_0), g(r_1, n_1)) \to |r(r_0, n_0) - r(g(r_1, n))|$.

Hence $|r(r_0, n_0) - r(g(r_1, n_1))| \geq |r(r_0, n_0) - r(r_1, n_1)| - \epsilon$. If r_1 large enough, this inequality can be written as

$$
r(g(r_1, n_1)) - r(r_0, n_0) \geq r(r_1, n_1) - r(r_0, n_0)
$$

and so

$$
r(g(r_1, n_1)) - r(r_1, n_1) \geq r(r_0, n_0) - r(r_0, n_0) - \epsilon.
$$

So $r \circ g - r$ is bounded from below by $-\epsilon = -d((r_0, n_0), g(r_0, n_0))$. Also

$r \circ g^{-1} - r$ is bounded from above by ϵ. So putting g in we get

$r \circ g^{-1}(g) - r(g) \leq \epsilon$. Hence $|r \circ g - r| \leq \epsilon$. ■

Lemma 4.1.17. *[30] Let $L^2(S) \oplus L^2(S)$ and $L^2(S \otimes E) \oplus L^2(S \otimes E)$ be even graded and $\Gamma_0(E) \oplus \Gamma_0(E)$ and $C_0(M) \oplus C_0(M)$ be odd graded and ρ be given by*

$$\rho : C_0(M) \oplus C_0(M) \longrightarrow \mathcal{L}(L^2(S) \oplus L^2(S))$$

$$f \oplus f \mapsto \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}, f \oplus -f \mapsto \begin{pmatrix} 0 & -if \\ if & 0 \end{pmatrix}. \tag{4.7}$$

there is an isomorphism between $(\Gamma_0(E) \oplus \Gamma_0(E)) \otimes_\rho (L^2(S) \oplus L^2(S))$ and $L^2(S \otimes E) \oplus L^2(S \otimes E)$ with even grading.

Proposition 4.1.18. *Let $\big[A\big] \in KK_G^{asymp}(C(N) \otimes C_1, C_0(M) \otimes C_1)$ be the cycle given by the triple in lemma 4.1.15, i.e.*

$$\big[A\big] = \big[\big(C_0(M)\hat{\otimes}C_1, e \otimes 1 : C(N)\hat{\otimes}C_1 \longrightarrow \mathcal{L}(\Gamma_0(E)\hat{\otimes}C_1), \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\big)\big] \tag{4.8}$$

and

$$\big[\bar{D}\big] := \big[\big(L^2(S\otimes E)\oplus L^2(S\otimes E), \phi, \begin{pmatrix} 0 & D_E \\ D_E & 0 \end{pmatrix}\big)\big] \in KK_G^{asymp}(C_0(M)\otimes C_1, \mathbb{C}), \tag{4.9}$$

where ϕ is given by multiplication. Suppose also that $D_E A - A D_E$ is bounded. Then we have

$$[A]\hat{\otimes}_{C_0(M)\hat{\otimes}C_1}[\bar{D}] = \tag{4.10}$$

$$\left[\left(L^2(S\otimes E)\oplus L^2(S\otimes E), e\otimes I, \begin{pmatrix} 0 & D_E + iI\otimes A \\ D_E - iI\otimes A & 0 \end{pmatrix}\right)\right]. \tag{4.11}$$

Proof. We have already shown that $[A]$ defines an equivariant cycle. That $[\bar{D}]$ defines an equivariant cycle is standard; a detailed proof is given in A.0.2 in the Appendix. We check the connection and positivity conditions.

To check the connection condition we need to show that the commutator $\left[\begin{pmatrix} D_p & 0 \\ 0 & \bar{D} \end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T_x^* & 0 \end{pmatrix}\right]$ is bounded for all x in some dense subset. We can confine our calculations to the odd and even cases by the linearity of T_x at x.

Using the isomorphism from

$$(\Gamma_0(E)\hat{\otimes}C_1)\otimes(L^2(S\otimes E)\oplus L^2(S\otimes E))$$

to

$$L^2(S\otimes E)\otimes C_1,$$

$T_x : L^2(S\otimes E)\oplus L^2(S\otimes E) \longrightarrow L^2(S\otimes E)\otimes C_1$ is given by

$$T_{f\oplus f}(S_M\oplus\overline{S_M}) := fS_M\oplus f\overline{S_M}.$$

Here we consider two cases: $x = (f, f)$, the even case, and $(f, -f)$, the odd case, by linearity of T in x, and use the standard notation that $A \otimes C_1$ shows $A \oplus A$ with odd grading. Then

$$
\begin{aligned}
(T_x \bar{D} - (-1)^{\partial x} D_p T_x) \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} &= T_x \bar{D} \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} - (-1)^{\partial x} D_p T_x \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} \\
&= (T_x \bar{D} - (-1)^{\partial x} D_p T_x) \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} \\
&= T_x \bar{D} \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} - (-1)^{\partial x} D_p T_x \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} \\
&= T_{f \oplus f} \begin{pmatrix} 0 & D_E \\ D_E & 0 \end{pmatrix} \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} \\
&- \begin{pmatrix} 0 & D_E + iI \otimes A \\ D_E - iI \otimes A & 0 \end{pmatrix} T_{f \oplus f} \begin{pmatrix} s_M \\ \overline{s_M} \end{pmatrix} \\
&= \qquad .
\end{aligned}
$$

.

.

$$T_{f\oplus f}\begin{pmatrix} D_E\bar{s_M} \\ D_E s_M \end{pmatrix} - \begin{pmatrix} 0 & D_E + iI\otimes A \\ D_E - iI\otimes A & 0 \end{pmatrix}\begin{pmatrix} fs_M \\ f\bar{s_M} \end{pmatrix}$$
$$= \begin{pmatrix} fD_E\bar{s_M} \\ fD_E s_M \end{pmatrix} - \begin{pmatrix} (D_E + iI\otimes A)(f\bar{s_M}) \\ (D_E - iI\otimes A)(fs_M) \end{pmatrix}$$
$$= \begin{pmatrix} fD_E\bar{s_M} \\ fD_E s_M \end{pmatrix} - \begin{pmatrix} D_E(f\bar{s_M}) \\ D_E(fs_M) \end{pmatrix} + i\begin{pmatrix} I\otimes A(f\bar{s_M}) \\ -I\otimes A(fs_M) \end{pmatrix}.$$

The term on the right is bounded since f can be chosen compactly supported.

As two differential operators $s_M \mapsto (fD_E\bar{s_M})$ and $s_M \mapsto (D_E f s_M)$ have the same symbol, their difference is a compact operator.

To check the boundedness of the commutator we need to simplify the expression $[F_1\hat{\otimes}Id, F]$ in which F_1 is the operator coming from the first cycle and F is the product module. More precisely we need, using the above isomorphism, to show that

$$\left[\begin{pmatrix} 0 & D_E + iI\otimes A \\ D_E - iI\otimes A & 0 \end{pmatrix}, (A\oplus -A)\otimes Id\right] \tag{4.12}$$

is bounded below. Using the above isomorphism the above commutator equals

$$2\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix} + i\begin{pmatrix} D_EA - AD_E & 0 \\ 0 & AD_E - D_EA \end{pmatrix}. \tag{4.13}$$

Now we note that by assumption $D_E A - AD_E$ is bounded and A^2 is arbitrary large outside some compact subset.

For the equivariance of the cycles, we note that Dirac operator is equivariant and A is equivariant at infinity. So $D_E + iI \otimes A$ is equivariant at infinity. For the compactness of the resolvent of D_p see [30, chapter 5].

■

Lemma 4.1.19. *$\psi : C(N) \otimes \mathbb{M}_n \otimes_e C_0(M) \longrightarrow C_0(M) \otimes \mathbb{M}_n$ defined by $f \otimes A \otimes_e g \mapsto e(f)g \otimes A$ is an isomorphism of $C_0(M)$-modules.*

Proof. We note that if $\{u_n\}$ is an approximate unit for $C_0(M)$, then the map $C_0(M) \otimes \mathbb{M}_n \longrightarrow C(N) \otimes \mathbb{M}_n \otimes_e C_0(M)$ defined by $f \otimes A \mapsto \lim_{n\to\infty}(f|_N \otimes A \otimes u_n)$ is a right inverse for ψ. So we only need to show that ψ preserves the inner products.

$\langle f_1 \otimes_e f_2, g_1 \otimes g_2\rangle = \langle g_2, e(\langle f_1, g_1\rangle)g_2\rangle = \langle f_2, e(f_1 g_1)g_2\rangle =$

$\langle f_2, e(\bar{f}_1)e(g_1)g_2\rangle =\rangle = \bar{f}_2\overline{e(f_1)}e(g_1)g_2$

On the other hand

$\langle\psi(f_1 \otimes_e f_2), \psi(g_1 \otimes_e g_2)\rangle = \langle e(f_1)f_2, e(g_1)g_2\rangle = \overline{e(f_1)f_2}e(g_1)g_2.$

■

Proposition 4.1.20. *Let $[S] \in KK(C_1, C(N)\hat{\otimes}C_1)$ be given by the triple*

$$\left(C(N)\hat{\otimes}C_1\otimes M_n, 1\otimes\psi\otimes 1 : C_1 \longrightarrow \mathcal{L}(C(N)\hat{\otimes}C_1\otimes M_n), \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} \otimes I_{\mathbb{M}_n}\right)$$

and

$[r] \in KK(C(N)\otimes\hat{C}_1, C_0M\hat{\otimes}C_1))$ *as before be given by*

$$\big(C_0(M)\hat{\otimes}C_1, e : C(N) \longrightarrow \mathcal{L}_{C_0(M)\hat{\otimes}C_1}(C_0(M)\hat{\otimes}C_1), \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\big),$$

then

$$[S]\otimes[r] = \big[\big(C_0(M)\hat{\otimes}C_1 \otimes \mathbb{M}_n, \psi \otimes_e 1, \begin{pmatrix} e(S)+r & 0 \\ 0 & -(e(S)+r) \end{pmatrix} \otimes I_{\mathbb{M}_n}\big)\big].$$

Proof. Under the isomorphism of $C(N)\otimes\mathbb{M}_n\otimes_e C_0(M)$ and $C_0(M)\otimes\mathbb{M}_n$, it is enough to check the connection and boundedness of the commutator and check the compactness of the resolvent. Under the above isomorphism $T_{f\otimes M\oplus f\otimes M}(f_1\oplus f_2) := e(f)f_1\otimes M \oplus e(f)f_2\otimes M$.

$$(T_{f\oplus f}\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} - \begin{pmatrix} e(S)+r & 0 \\ 0 & -(e(S)+r) \end{pmatrix} T_{f\oplus f})\begin{pmatrix} g \\ h \end{pmatrix}$$

$$= T_{f\oplus f}\begin{pmatrix} rg \\ rh \end{pmatrix} - \begin{pmatrix} e(S)+r & 0 \\ 0 & -(e(S)+r) \end{pmatrix}\begin{pmatrix} e(f)g \\ e(f)h \end{pmatrix}$$

$$= \begin{pmatrix} e(f)rg \\ e(f)rh \end{pmatrix} - \begin{pmatrix} (e(S)+r)(e(f)g) \\ -(e(S)+r)(e(f)h) \end{pmatrix}$$

$$= \begin{pmatrix} (e(f)rg - r(e(f)g) \\ e(f)rh - r(e(f)h)) \end{pmatrix} - \begin{pmatrix} e(S)(e(f)g) \\ -e(S)(e(f)h) \end{pmatrix}.$$

Now, the connection condition is obvious.

Checking the boundedness of the commutator is similar to 4.13. ■

Convention on notation To simplify the notation, from now on we set $\overline{M} := \overline{[0,\infty) \times \mathbb{S}^{n-1}}$[2], where $M := [0,\infty) \times \mathbb{S}^{n-1}$ is equipped with metric $dr^2 + sinh^2(r)d_{\mathbb{S}^{n-1}}$, and $N := S^{n-1}$.

Remember that as an example we showed[3]that Poincaré disk has the structure of a warped cone. One can show in the same way that the ball model for n-dimensional manifold has the structure of a warped-cone. One can see the $n-$dimensional ball model as $[0,\infty) \times S^{n-1}$. After perturbing the Riemannian metric and compactification, as was explained in the previous chapter, one gets $\overline{M} := [0,\infty] \times S^{n-1}$, a manifold with boundary S^{n-1}. Geometrically, the boundary consists of ideal points. Fix an ideal point called ∞ in the ball model. Now for every point on the boundary $s \in S^{n-1}$ and every function $f \in C(S^{n-1})$, define f on the line connecting s and ∞ by the constant value $f(s)$.

r is the multiplication operator by radius.

We need to simplify the module and operators as much as possible. To do so, we prove the following lemma.

Lemma 4.1.21. *$C(\overline{M}) \otimes L^2(\overline{M}) \cong L^2(\overline{M})$ as $C(\overline{M})$-modules.*

[2]bar stands for closure

[3]see the appendix A.0.4

Proof. Let $\phi : C(\overline{M}) \otimes L^2(\overline{M}) \longrightarrow L^2(\overline{M})$ be given by the rule $(f,g) \mapsto fg$. ϕ is an isomorphism of $C(\overline{M})-$modules with the inverse given by

$$\psi : L^2(\overline{M}) \longrightarrow C(\overline{M}) \otimes L^2(\overline{M})$$

$$f \mapsto 1 \otimes f.$$

Now suppose $\langle f \otimes g, f \otimes g \rangle = 0$. Then using the definition of the inner product on $C(\overline{M}) \otimes L^2(\overline{M})$, we have:

$$\langle f \otimes g, f \otimes g \rangle = \langle g, \phi_2(\langle f, f \rangle) g \rangle = \int fg(fg)^* = 0$$

where ϕ_2 is the operator coming from the second cycle and is given by multiplication. Now remember the fact from functional analysis that if the integral of a non-negative function is zero, then that function should be zero almost every where. So $fg = 0$ *a.e.*

By linearity of ψ we have then $1 \otimes fg = f \otimes g = 0$. So in the construction of $C(\overline{M}) \otimes L^2(\overline{M})$ no non-zero element is eliminated, and the above isomorphism is in fact a $C(\overline{M})$ isomorphism of Hilbert $C(\overline{M})$-modules. ■

Definition 4.1.22. *[24] Two Kasparov $A-B$ modules (X_0, F_0) and (X_1, F_1) are called unitarily equivalent if there exist an even unitary $U : X_0 \longrightarrow X_1$ intertwining F_0 and F_1 and ϕ_0 and ϕ_1, i.e.*

$$UF_0U^* = F_1, \text{ and } U\phi_0(a)U^* = \phi_1(a), \forall a \in A.$$

Let $\mathbf{ev_t}\colon C([0,1],B) \longrightarrow B, f \mapsto f(t)$, be the evaluation at $t \in [0,1]$. Remember that under isomorphism $C([0,1],B) \cong C([0,1]) \otimes B$ we have the following definition.

Definition 4.1.23. *[24] Two Kasparov modules (X_0,F_0) and (X_1,F_1) are called **homotopic** if there exists an $A-C([0,1],B)$ Kasparov module (X,F) such that*

$$(X\hat{\otimes}_{ev_0}B, F\hat{\otimes}_{ev_0}1) \sim_u (X_0,F_0) \text{ and } (X\hat{\otimes}_{ev_0}B, F\hat{\otimes}_{ev_1}1) \sim_u (X_1,F_1).$$

In the next proposition we are going to find the Kasparov product of two cycles

$$\left[\bar{r}\right] = \left[\left(C(\bar{M})\hat{\otimes}C_1, \bar{e}\hat{\otimes}C_1 : C(N) \longrightarrow L(L^2(\bar{M})\hat{\otimes}C_1), r\otimes e_1\right)\right]$$

and

$$\left[\overline{D_{\bar{M}}}\right] = \left[\left(L^2(\bar{M})\hat{\otimes}C_1, \phi_2 : C(\bar{M})\hat{\otimes}C_1 \longrightarrow L(L^2(\bar{M})\hat{\otimes}C_1), \begin{pmatrix} 0 & \overline{D_{\bar{M}}} \\ \overline{D_{\bar{M}}} & 0 \end{pmatrix}\right)\right],$$

where e is defined by multiplication by the extended map.

Proposition 4.1.24. $\left[\bar{r}\right] \otimes \left[\overline{D_{\overline{M}}}\right] \cong \left[D_N\right] \in KK(C(N)\hat{\otimes}C_1, \mathbb{C})$.

Proof. Exactly the same as lemma 4.1.18 one can find the product

$$[\bar{r}] \otimes [\overline{D_{\overline{M}}}].$$

In fact if we consider just one copy of $C(\overline{M})$, then, using the mentioned lemma, the product

$[(C(\overline{M}), \bar{e} : C(N) \longrightarrow L(C(\bar{M})), r)] \otimes [(L^2(\overline{M}) \oplus (L^2(\overline{M}), \phi_2 : C(\overline{M}) \longrightarrow L(L^2(\overline{M}) \oplus L^2(\overline{M})), \begin{pmatrix} 0 & \overline{D_{\overline{M}}} \\ \overline{D_{\overline{M}}} & 0 \end{pmatrix})]$ is given by

$$[(L^2(\overline{M}) \oplus L^2(\overline{M}), 1 \otimes \bar{e}, \begin{pmatrix} 0 & \overline{D_{\overline{M}}} + ir \\ \overline{D_{\overline{M}}} + ir & 0 \end{pmatrix})].$$

What remains to show is that the above product cycle is equivalent to D_N.

We will construct a Kasparov homotopy of two cycles in $KK(C(N), \mathbb{C})$. The homotopy will be given by a cycle in $KK(C(N), C([0,1]))$.

The module will be given by $E := \{f \in C([0,1], L^2(\overline{M})); f(1) \in L^2(N)\}$, and the cycle is given by $\mathcal{E} = (E, \phi : C(N) \longrightarrow \mathcal{L}_{\mathbb{C}}(E), D_N + (1-\lambda)i(\partial_r + r)) \in \mathbb{E}(C(N) \otimes \mathbb{C}, C([0,1]) \otimes \mathbb{C})$. We note that here we are using dimension drop method. In example 2.0.2 we observed that E with operations $(g.f)(x) := g(x)f(x)$ and $\langle f, g\rangle_{C([0,1])}(x) := \langle f(x), g(x)\rangle_{L^2(\overline{M})}$ is a $C([0,1])-$module.

We now check the boundedness property. As $\phi(f)$ commutes with r, we have $[\phi(f), D_N + (1-t)i(\partial_r + r)] = [\phi(f), D_N + (1-t)i(\partial_r)]$. At t=0 we get the original cycle back . For other steps, we look at this as an equation with two coordinates, coordinate in direction N and the one in direction r. The norm in r direction is getting smaller and smaller when t increases.

For the compactness of the resolvent, we note again that the first step has compact resolvent and the steps after the resolvent remains compact because multiplication by t of $\partial r + r$ is the same as changing the metric and using the same operator $\partial r + r$.

Last we use the sequence

$$L^2(\overline{M}, E) \underset{s_*}{\overset{p_*}{\rightleftharpoons}} L^2(N, i^*E) \longrightarrow 0$$

to project $L^2(\overline{M})$ onto $L^2(N)$. We note that as $p_*(\phi(f)s) = p_*(\phi(f))p_*(s)$, $p_*(\phi(f))$ is just multiplication f.

Finally, we state and prove the main result of this chapter, the calculation of the index of an approximately equivariant Dirac-Schrödinger operator.

Theorem 4.1.25. *Let $D + iV$ be a Dirac-Schrödinger operator whose potential V satisfies the conditions of lemma 4.1.15. Then*

$$[D_M + iV] = [S] \otimes [D_N] \tag{4.14}$$

in which $[S]$ is the cycle introduced in proposition 4.1.20.

Proof. In proposition 4.1.18 we showed that $[D_M + iV] = [V] \otimes [D_M]$ and in 4.1.20 we proved that $[V] = [S] \otimes [r]$.

By associativity of the product and the above two products we have $[D_M + iV] = [V] \otimes [D_M] = ([S] \otimes [r]) \otimes [D_M] = [S] \otimes ([r] \otimes [D_M])$.

Using the homotopy in proposition 4.1.24 we have $[r] \otimes [D_M] = [D_N]$.

So $[D_M + iV] = [S] \otimes [D_N]$. ■

Corollary 4.1.26. $\text{Index}_G D + iV = \int \hat{A}(N) \wedge ch(E_S^+)$.

Proof. Atiyah−Singer index theorem calculates the right hand of 4.1.1 as the integral $\int \hat{A}(N) \wedge ch(E_S^+)$. ■

There was the Door
I found no key;
There was the Veil
Through which
I might not see
Khayyam

Appendix A

Appendix

A.0.1 Explicit isomorphisms of some warped-cones

Example A.0.1. $\mathbb{R}^2\backslash 0$ *has the structure of a warped-product. Let the metric on* $(0,\infty)\times S^1$ *be given by* $r^2 d\theta\otimes d\theta+dr\otimes dr$. *We will calculate the pull-back of the Euclidean metric under the following diffeomorphism*

$$F:(0,\infty)\times\mathbb{S}^1\longrightarrow\mathbb{R}^2\setminus\{0\}$$

$$(r,\theta)\mapsto(r\cos\theta, r\sin\theta)$$

and show that the pulled back metric we find is in fact $r^2 d\theta\otimes d\theta+dr\otimes dr$; *this shows that* F *is an isometry.*

From the properties of pull-back we have:$F^*(dx^2+dy^2)=F^*(dx^2)+F^*(dy^2)$, $F^*(dx^2)=F^*(dx\otimes dx)=F^*(dx)\otimes F^*(dx)$, $F^*(dy^2)=F^*(dy\otimes dy)=$

$F^*(dy) \otimes F^*(dy)$, but $F^*(dx) = d(F^*x), F^*(dy) = d(F^*y)$. So

$$F^*(dx^2 + dy^2) = F^*(dx^2) + F^*(dy^2) = d(F^*x) \otimes d(F^*x) + d(F^*y) \otimes d(F^*y).$$

So it is enough to find $d(F^*x)$ *and* $d(F^*y)$.

$$F^*(dx) = d(F^*x) = d(x \circ F) = d(r\cos\theta) = \cos(\theta)dr - r\sin(\theta)d\theta$$

$$F^*(dy) = d(F^*y) = d(y \circ F) = d(r\sin(\theta) =) \sin(\theta)dr + r\cos(\theta)d\theta$$

$$F^*(dx) \otimes F^*(dx) = (\cos(\theta)dr - r\sin(\theta)d\theta) \otimes (\cos(\theta)dr - r\sin(\theta)d\theta) =$$

$\cos^2(\theta)dr \otimes dr - r\cos(\theta)\sin(\theta)dr \otimes d\theta - r\sin(\theta)\cos(\theta)d\theta \otimes dr + r^2\sin^2(\theta)d\theta \otimes d\theta$,

by linearity of tensor fields on smooth functions and properties of pull-back. in the same way,

$$F^*(dy) \otimes F^*(dy) = (\sin\theta dr + r\cos\theta d\theta) \otimes (\sin\theta dr + r\cos\theta d\theta) =$$

$\sin^2(\theta)dr \otimes dr + r\sin(\theta)\cos(\theta)dr \otimes d\theta + r\sin(\theta)\cos(\theta)d\theta dr + r^2\cos^2(\theta)d\theta \otimes d\theta$.

Hence

$F^*(dx^2 + dy^2) = F^*(dx^2) + F^*(dy^2) = \cos^2(\theta)dr \otimes dr - r\cos(\theta)\sin(\theta)dr \otimes d\theta - r\sin(\theta)\cos(\theta)d\theta \otimes dr + r^2\sin^2(\theta)d\theta \otimes d\theta + \sin^2(\theta)dr \otimes dr + r^2 d\theta \otimes d\theta + \sin^2(\theta)dr \otimes dr + r\sin(\theta)\cos(\theta)dr \otimes d\theta + r\sin(\theta)\cos(\theta)d\theta \otimes dr + r^2\cos^2(\theta)d\theta \otimes$

$d\theta^2 = r^2 d\theta \otimes d\theta + dr \otimes dr$. It is obvious that F is a smooth function having a smooth inverse. So it is a diffeomorphism, which preserves metric and hence an isometry.

Example A.0.2. *[37] $\mathbb{R}^3 \setminus 0$ as warped product: In the spherical coordinate, the line element of $\mathbb{R}^3 \setminus 0$ is given by*

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

Setting $r = 1$ gives the line element of the unit sphere $\mathbb{S}^2$. Evidently, $\mathbb{R}^3 \setminus 0$ is diffeomorphic to $\mathbb{R}^+ \times \mathbb{S}^2$ under the natural map $(t, p) \mapsto tp$. Thus the formula for ds^2 shows that $\mathbb{R}^3 \setminus 0$ can be identified with the warped product $\mathbb{R}^+ \times \mathbb{S}^2$.

In the following example, the same as example A.0.1, we give another proof using pulling back of the metric.

Example A.0.3. *Warped cone structure on $\mathbb{R}^3 \setminus 0$ is given by pulling back of the flat metric under a diffeomorphism on $(0, 2\pi) \times (0, \pi) \times \mathbb{R}^+$. In fact we consider $(0, 2\pi) \times (0, \pi) \times \mathbb{R}^+$ with the Riemannian metric $d\rho^2 + \rho^2 d\varphi^2 + \rho^2 \sin^2(\phi)d\theta^2$ and show that pulling back of the flat metric on $\mathbb{R}^3 \setminus \{0\}$ results in the same metric.*

Define $F : (0, 2\pi) \times (0, \pi) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 \setminus 0$ by

$$F(\theta, \varphi, \rho) := \big(\rho \sin(\varphi) \cos(\theta), \rho \sin(\varphi) \sin(\theta), \rho \cos(\varphi)\big).$$

$F^*(dx) = d(x \circ F) = d(\rho \sin(\varphi) \cos(\theta)) = \sin(\varphi) \cos(\theta) d\rho + \rho \cos(\varphi) \cos(\theta) d\varphi -$

$\rho\sin(\varphi)\sin(\theta)d\theta$.

$F^*(dy) = d(y\circ F) = d(\rho\sin(\varphi)\sin(\theta)) = \sin(\varphi)\sin(\theta)d\rho+\rho\cos(\varphi)\sin(\theta)d\varphi+\rho\sin(\varphi)\cos(\theta)d\theta$.

$F^*(dz) = d(z\circ F) = d(\rho\cos(\varphi)) = \cos(\varphi)d\rho - \rho\sin(\varphi)d\varphi$.

From the above equations we have:

$F^*(dx^2) = d(x\circ F)\otimes d(x\circ F) = \sin^2(\varphi)\cos^2(\theta)d\rho\otimes d\rho+\rho\sin\varphi cos^2\cos\varphi d\rho\otimes d\varphi - \sin^2\varphi\cos\theta\sin\theta d\rho\otimes d\theta+\rho\cos\varphi\sin\varphi\cos^2\theta d\varphi\otimes d\rho+\rho^2\cos^2\varphi\cos^2\theta d\varphi\otimes \varphi - \rho^2\cos\varphi\cos\theta\sin\varphi\sin\theta d\varphi\otimes d\theta+\rho^2\sin^2\varphi\sin\theta\cos\theta d\theta\otimes d\rho+\rho^2\sin\theta\sin\varphi\cos\theta\cos\varphi d\theta\otimes d\rho + \rho^2\sin^2\theta\sin^2\varphi d\theta\otimes d\theta$.

$F^*(dy^2) = d(y\circ F)\otimes d(y\circ F) = \sin^2(\varphi)\sin^2(\theta)d\rho\otimes d\rho+\rho\sin\varphi\cos\varphi sin^2\theta d\rho\otimes d\varphi+\rho^2\cos^2(\varphi)\sin^2(\theta)d\varphi\otimes d\varphi+\rho sin\theta\sin^2\varphi\cos\theta d\rho\otimes d\theta+\rho\cos\varphi sin\varphi\sin^2\theta d\varphi\otimes d\rho+\rho^2\cos\varphi\sin\varphi\sin\theta\cos\theta d\varphi\otimes d\theta+\rho\sin^2\varphi\cos^2\theta d\theta\otimes d\rho+\rho^2\sin\varphi\cos\varphi\cos\theta\sin\theta d\theta\otimes d\varphi + \rho^2\sin^2\varphi\cos^2\theta d\theta\otimes d\theta$.

$F^*(dz^2) = d(z\circ F)\otimes d(z\circ F) = \cos^2(\varphi)d\rho\otimes d\rho + \rho^2\sin^2(\varphi)d\varphi\otimes\varphi - \rho\cos\varphi\sin\varphi d\varphi\otimes d\varphi - \rho\cos\varphi\sin\varphi d\varphi\otimes d\rho$.

Using the above identities and the basic properties of trigonometric functions, one can check that $F^*(dx^2)+F^*(dy^2)+F^*(dz^2) =$

$$d\rho^2+\rho^2d\varphi^2+\rho^2\sin^2(\phi)d\theta^2.$$

Example A.0.4. $\mathbb{D} := \{(x,y)\in\mathbb{R}^2 |\ x^2+y^2<1\}$ *with the hyperbolic metric* $\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$ *has a warped cone structure.*

We define $F : [0,1)\times\mathbb{S}^1 \longrightarrow \mathbb{D}$ by $F(r,\theta) := (r\cos(\theta), r\sin(\theta))$. $\mathbb{D}$

already has a Riemannian metric. So it is enough to pull it back by F^*. So we compute $F^*(\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2})$.

$F^*(\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}) = \frac{4}{(1-x^2-y^2)^2} \circ F.F^*(dx^2+dy^2)$ From the previous example we know that $F^*(dx^2 + dy^2) = dr^2 + r^2 d\theta^2$. So, $\frac{4}{(1-x^2-y^2)^2} \circ F.F^*(dx^2 + dy^2) = \frac{4}{(1-(r\cos\theta)^2-(r\sin\theta)^2)^2}.F^*(dx^2+dy^2) = \frac{4}{(1-(r\cos\theta)^2-(r\sin\theta)^2)^2}.(dr^2+r^2d\theta^2) = \frac{4}{(1-r^2)^2}.(dr^2 + r^2d\theta^2) = \frac{4}{(1-r^2)^2}dr^2 + \frac{4}{(1-r^2)^2}r^2d\theta^2$. The same calculation as the previous examples shows that the metric is given by $dr^2 + \frac{4}{(1-r^2)^2}r^2d\theta^2$.[1].

A.0.2 Equivariance of cycles

Recall that a group G acts on a C^*-algebra B if there is a $*$-homomorphism $\alpha : G \longrightarrow Aut(B)$, of zero degree operators! A group G acts on a $B-$Hilbert module E if there is a homomorphism, also denoted by α from G to the set of invertable bounded operators on E, and $\alpha_g(eb) = \alpha_g(e)\alpha_g(b), \quad and \quad \alpha_g < x, y > = < \alpha_g x, \alpha_g y >$.

Suppose G is a discrete group, acting isometrically on a complete Riemannian manifold $(M.g)$, and (S, ∇^S) is an invariant Dirac bundle on M. Let D_M be the resulting Dirac operator on $L^2(M, S)$.

$C_0(M)$ defines a G-C^*-algebra for the action $\alpha : G \longrightarrow Aut(C_0(M))$.

The only thing that should be checked is strong continuity of the action which is obvious because every map with discrete domain is continuous.

[1]One can also take a loot at [13]

The corresponding action on S is given by the unitary representation

$$U : G \longrightarrow U(L^2(M, S))$$

$$\forall g \in G, \forall \xi \in C(M, S), \forall x \in M, U_g(\xi)(x) := g.\xi(g^{-1}x).$$

Let $g \in G$. U_g is isometric:

$$\langle U_g\xi, U_g\eta\rangle_B = \langle \xi, \eta\rangle_B.$$

In fact

$$\langle U_g\xi, U_g\eta\rangle = \int \langle U_g\xi(x), U_g\eta(x)\rangle dx = \int \langle g\xi(g^{-1}x), g\eta(g^{-1}x)\rangle_x dx =$$

$$\int \langle \xi(g^{-1}x), \eta(g^{-1}x)\rangle_{g^{-1}x} = \int \langle \xi(x), \eta(x)\rangle dx = \langle \xi, \eta\rangle.$$

In the third equality we have used isometry of the action and in the fourth one the substitution $g^{-1}x = x$; U_g is invertible since $U_{g^{-1}} \circ U_g = U_g \circ U_g^{-1} = U_{e_G} = Id$.

Now let $\phi : C_0(M) \longrightarrow \mathcal{L}_{\mathbb{C}}(L^2(M, S))$ be the action given by left multiplication. We need also to show that

$$\forall g \in G, \forall f \in C_0(M) U_g\phi(f)U_{g^{-1}} = \phi(g.f).$$

For this we note that

$$\forall \sigma \in L^2(M,S), U_g\phi(f)U_{g^{-1}}(\sigma) = U_g f.(\sigma \circ g) = (f \circ g^{-1}).\sigma = \phi(g.f)\sigma.$$

We now check the strong G- continuity of D.

By definition we have $D\sigma = \Sigma_{j=1}^{nc}(e_j).\nabla_{e_j}^{\sigma}$. We note that $gc(\nu_x)\sigma_x = c(g\nu_x)g.\sigma_x$. So, pointwise, we have $U_g(c(\nu)\sigma) = c(g\nu)U_g\sigma$.

Hence, $U_g c(X) U_{g^{-1}} = c(g_x X)$.So the bundle S, and connection ∇ are invariant under the action of g, so $D_M g - gD_M = 0$, which means that D_M is g continuous.

Next we show that $(L^2(M,S), \phi, [D_M])$ defines a cycle in

$KK^G_{\dim M}(C_0(M), \mathbb{C})$.

We need to show that $\phi(a)(i+D)^{-1}$, and $(i+D)^{-1}\phi(a)$ are compact operators, for all a in a dense subset of $L^2(M,S)$. So it suffices to check this, compactness, for all $a \in C_0(M,S)$. In fact it is enough to confine ourselves to compactly supported smooth functions that are dense in $C_0(M,S)$. Suppose Ω is a bounded open set containing the support of a. We note that if $a \in C_c^{\infty}(\Omega)$, then $\phi(a)$ is a bounded operator from $H^1(M)$ to $H^1(\Omega)$, $\phi(a) \in B(H^1(M), H^1(\Omega))$, and that by the Rellich- Kondrachov lemma the inclusion $H^1(\Omega, S) \longrightarrow L^2(\Omega, S)$ is compact. We also note that D is essentially self-adjoint and has a bounded extension $(D+i)^{-1} : L^2(M,S) \longrightarrow H^1(M,S)$.

Thus we have the following sequence of bounded operators:

$$L^2(M,S) \xrightarrow[\text{bounded}]{(D+i)^{-1}} H^1(M,S) \xrightarrow[\text{bounded}]{\phi(a)} H^1(\Omega,S) \xrightarrow{\text{Rellich-Kondrachov embedding}}$$

$$L^2(\Omega,S) \hookrightarrow L^2(M,S)$$

So, $\hookrightarrow \phi(a)(D+i)^{-1}$ is compact (As inclusion is compact, and set of compact operators is an ideal). So $\phi(a)(D+i)^{-1}$ is compact.

A.0.3 Explicit calculation of Dirac operator on the plane

We just proved that $\mathbb{R}^2 \setminus 0$ is a warped-cone. Now we will find the Dirac operator on this space.

Direct calculation shows that the Dirac operator on $\mathbb{R}^2 \setminus \{0\}$ is given by $\begin{pmatrix} 0 & i\partial_x + \partial_y \\ -i\partial_x - \partial_y & 0 \end{pmatrix}$. Let $\mathcal{D} = a\partial_x + b\partial_y$. We want to find a, b in such a way that $\mathcal{D}^2 = -\frac{\partial^2}{\partial_x^2} - \frac{\partial}{\partial_y^2}$. So $\mathcal{D}^2 = (a\frac{\partial}{\partial_x^2} + b\frac{\partial}{\partial_y^2})(a\frac{\partial}{\partial_x^2} + b\frac{\partial}{\partial_y^2}) = a^2\frac{\partial}{\partial_x^2} + (ab+ba)\frac{\partial^2}{\partial x \partial y} + b\frac{\partial}{\partial_y^2}$. So we need to solve

$$\begin{cases} a^2 = -1 \\ ab + ba = 0 \\ b^2 = -1. \end{cases} \tag{A.1}$$

A solution to this equation is $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So $\mathcal{D}$ is given by

$\begin{pmatrix} 0 & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & 0 \end{pmatrix}$. Using chain rule for vector fields

$$\frac{\partial}{\partial x} = \frac{\partial_r}{\partial_x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}$$

using $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, we have

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = x/r = cos\theta, \quad \frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = y/r = sin\theta$$

and

$$\frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = \frac{-1}{r}sin\theta, \quad \frac{\partial\theta}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{1}{r}cos\theta$$

So

$$\frac{\partial}{\partial x} = cos\theta\frac{\partial}{\partial r} + \frac{-1}{r}sin\theta\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = sin\theta\frac{\partial}{\partial r} + \frac{1}{r}cos\theta\frac{\partial}{\partial\theta}$$

Hence the Dirac operator in polar coordinate can be written as

$$\begin{pmatrix} 0 & i(cos\theta\frac{\partial}{\partial r} + \frac{-1}{r}sin\theta\frac{\partial}{\partial\theta}) + sin\theta\frac{\partial}{\partial r} + \frac{1}{r}cos\theta\frac{\partial}{\partial\theta} \\ -i(cos\theta\frac{\partial}{\partial r} + \frac{-1}{r}sin\theta\frac{\partial}{\partial\theta}) - sin\theta\frac{\partial}{\partial r} - \frac{1}{r}cos\theta\frac{\partial}{\partial\theta} & 0 \end{pmatrix} =$$

$$
\begin{aligned}
&(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} cos\theta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} sin\theta)\frac{\partial}{\partial r} + ((\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \frac{-1}{rsin\theta} \\
&\quad + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{r}cos\theta))\frac{\partial}{\partial \theta} \\
&\quad = (cos\theta\frac{\partial}{\partial x} + sin\theta\frac{\partial}{\partial y})\frac{\partial}{\partial r} + (\frac{-1}{r}sin\theta\frac{\partial}{\partial x} + \frac{1}{r}cos\theta\frac{\partial}{\partial y})\frac{\partial}{\partial \theta} \\
&\quad = ((cos\theta\frac{\partial}{\partial r} - \frac{-1}{r}sin\theta\frac{\partial}{\partial \theta})cos\theta + (sin\theta\frac{\partial}{\partial r} + \frac{1}{r}cos\theta\frac{\partial}{\partial \theta}))\frac{\partial}{\partial r} \\
&\quad + (\frac{-1}{r}sin\theta cos\theta\frac{\partial}{\partial r} + \frac{1}{r^2}sin^2\theta\frac{\partial}{\partial \theta} + \frac{1}{r}sin\theta cso\theta\frac{\partial}{\partial r} \\
&\quad + \frac{1}{r^2}cos^2\theta\frac{\partial}{\partial \theta})\frac{\partial}{\partial \theta} \\
&\quad = ((1)\frac{\partial}{\partial r})\frac{\partial}{\partial r} + (\frac{1}{r^2}\frac{\partial}{\partial \theta})\frac{\partial}{\partial \theta} \\
&\quad = (\frac{\partial}{\partial r})\frac{\partial}{\partial r} + ((-i)\frac{1}{r}\frac{\partial}{\partial \theta})(-i)\frac{1}{r}\frac{\partial}{\partial \theta} \\
&\quad = c(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + c(\frac{1}{r}\frac{\partial}{\partial \theta})(-i)\frac{1}{r}\frac{\partial}{\partial \theta},
\end{aligned}
$$

in which $(-i)\frac{1}{r}\frac{\partial}{\partial\theta}$ is the Dirac operator on the circle of radius r and $c(\frac{1}{r}\frac{\partial}{\partial\theta})$ is the corresponding Clifford action, and $\frac{\partial}{\partial r}$ is the Dirac operator on the real line and $c(\frac{\partial}{\partial r})$ is the corresponding Clifford multiplication. We note that in the above calculation we have used the following identification implicitly:

$$c : \mathfrak{X}(\mathbb{R}^2) \longrightarrow \Gamma(End(\mathbb{R}^2) \times \mathbb{C}^2)$$

$$\frac{\partial}{\partial x} \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \text{ and } \frac{\partial}{\partial x} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A.0.4 A Kasparov product when the group action is compact

Finally, we use a result in [7] to calculate a Kasparov product in the case of compact group action. We note that an application of the following lemma by Baum calculates a similar Kasparov product for the case of the action of compact Lie groups[2] on hyperbolic manifolds.

Lemma A.0.5. *[7, Lemma B.10] Let L be a G- spin $\mathbb{C}$ manifold with boundary ∂L, let M be a G-invariant sub-manifold of ∂L with boundary ∂M such that* $\dim M = \dim L - 1$. *Let $[L \setminus \partial L]$ and $[M \setminus \partial M]$ be the associated cycles to manifolds $M \setminus \partial M$ and $L \setminus \partial L$, and let $\partial \in KK^G(C_0(M \setminus \partial M), C_0(L \setminus \partial L))$ be the boundary element associated to the exact sequence*

$$0 \longrightarrow C_0(L \setminus \partial L) \longrightarrow C_0((L \setminus \partial L) \cup (M \setminus \partial M)) \longrightarrow C_0(M \setminus \partial M) \longrightarrow 0.$$

Then $[\partial] \otimes [L \setminus \partial L] = [M \setminus \partial M]$.

In order to use this result for hyperbolic manifolds, let's take a look at

[2]Assumption of amenability is not needed for abelian compact groups as these groups are amenable.

the closure of manifold

$$[0,\infty)\times\mathbb{S}^{n-1} \;\; with\ metric\ dr^2+sinh^2(r)d_{\mathbb{S}^{n-1}}.$$

This is a manifold with boundary $\mathbb{S}^{n-1}$.

If we use the notation of the above lemma and set $L := \overline{[0,\infty)\times\mathbb{S}^{n-1}}$ and $M := \mathbb{S}^{n-1}$, then we get the following result.

$$0\longrightarrow C_0(\overline{[0,\infty)\times\mathbb{S}^{n-1}}\setminus\mathbb{S}^{n-1})\longrightarrow C_0((\overline{[0,\infty)\times\mathbb{S}^{n-1}}\setminus\mathbb{S}^{n-1})\cup(\mathbb{S}^{n-1}\setminus\emptyset))$$

$$\longrightarrow C_0(\mathbb{S}^{n-1}\setminus\emptyset)\longrightarrow 0$$

which is the same as the following exact sequence

$$0\longrightarrow C_0([0,\infty)\times\mathbb{S}^{n-1})\longrightarrow C(([0,\infty]\times\mathbb{S}^{n-1}))\longrightarrow C(\mathbb{S}^{n-1})\longrightarrow 0.$$

So if we let G be a compact Lie group acting by isometries, then we have

$[\partial]\otimes[[0,\infty)\times\mathbb{S}^{n-1}]=[\mathbb{S}^{n-1}]$.

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