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by

U.G. Gujar, V.C. Bhavsar and N. Vangala

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Faculty of Computer Science
University of New Brunswick
P.O. Box 4400
Fredericton, N.B. E3B 5A3

Phone: (506) 453-4566
Fax: (506) 453-3566

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Uday G. Gujar
Virendra C. Bhavsar
Nagarjuna Vangala

Faculty of Computer Science
University of New Brunswick
Fredericton, N.B.
CANADA E3B 5A3

Abstract

The transformation function $z \leftarrow z^{\alpha} + c$ is used for generating fractal images in the complex z-plane. When α is a positive integer the fractal image has a lobular structure with α major lobes. When α is a negative integer the image has a planetary configuration consisting of a central planet with $|\alpha|$ major satellite structures. For non-integer values of α, additional embryonic lobular/satellite structures, proportional in size to the fractional part of α, are observed. Based on the extensive experimentation, six conjectures regarding the number of major as well as embryonic lobular/satellite structures, their positions and angular spaces are formulated.

1. Introduction

The self-squared function, $z \leftarrow z^2 + c$, for generating fractals has been discussed extensively in the literature [MAND82, PEIT86]. We have proposed the transformation function $z \leftarrow z^\alpha + c$ for generating fractal images in the complex c-plane and z-plane for positive and negative integer as well as non-integer values of α [KALR88, GUJA88, VANG89]. Several conjectures regarding the visual characteristics of the c-plane images have been given in [GUJA88] while the z-plane images are discussed in [VANG89]. We were made aware of the work reported by Lakhtakia et al., on the generation of fractal images from $z \leftarrow z^\alpha + c$ for the positive integer values of α [LAKH87], when the report [GUJA88] was submitted for publication (this report, in the revised form appears in [GUJA90]).

In this paper we consider the z-plane fractal images from $z \leftarrow z^\alpha + c$. Lakhtakia et al. [LAKH87] have given few images for positive integer values of α. We have verified this work and extended it to positive non-integer values of α. In addition, we experiment with the negative integer as well as non-integer values of α. Finally, we formulate several conjectures about the structure of the z-plane fractal images.

2. Preliminaries

A fractal image consists of a two-dimensional array of pixels. Each pixel is represented by a pair of (x,y) co-ordinates. For the z-plane fractal images (which are referred to as Julia Sets in the literature for positive integer α values [PEIT86,LAKH87]) the (x,y) coordinates of each pixel are associated with the real and imaginary parts of z, i.e., (z_x,z_y). For each pixel, the value of z is computed iteratively. Two criteria are used to terminate this iterative process:

1. the value of z diverges beyond a certain preset limit L, or
2. the allowable number of iterations, N, is reached.

If the iterations terminate due to the condition (1), then the transformation function for that pixel is unstable, while if these terminate due to the condition (2), then the function is stable at that pixel. We have discussed this process of generating images at length in [GUJA90].

The exponent α can be represented as $\alpha = \pm(\eta + \varepsilon)$, where η is a positive integer and ε is the fractional part, $0 \leq \varepsilon < 1$. The fractal images generated from the transformation function $z \leftarrow z^{\alpha} + c$ in the complex z-plane with c as a constant are denoted in this paper by $\wp^z$. Based on this notation, we classify the z-plane fractal images as follows:

1. $\{\wp^z | \alpha = +\eta\}$: the set of fractal images for positive integer α,
2. $\{\wp^z | \alpha = +(\eta + \varepsilon)\}$: the set of fractal images for positive non-integer α,
3. $\{\wp^z | \alpha = -\eta\}$: the set of fractal images for negative integer α,
4. $\{\wp^z | \alpha = -(\eta + \varepsilon)\}$: the set of fractal images for negative non-integer α.

For all the images appearing in this paper, the z-plane region with the lower left corner of (-1.5, -1.5) and the upper right corner of (1.5, 1.5) is chosen. The initial value of c, c_0, is chosen as $c_0 = 0.5 + i\ 0.5$. The maximum number of iterations, N, is set to 100 and the divergence limit, L, for the magnitude of z is set to 10.

3. Positive Integer α

Several fractal images for the positive integer values of α have been generated. Figure 1 contains two images for $\alpha = 5$ and $\alpha = 12$. The image in Fig. 1(a) consists of five self-similar major lobes which are oriented with angular symmetry around the origin. The stable region, represented in white, is surrounded by the unstable region. Twelve self-similar major lobes are seen in Fig. 1(b) for $\alpha = 12$. Lakhtakia et al. [LAKH87] have presented images for $\alpha = 2$, 3, 4 and 5, for various values of c and identified this α-fold symmetry. It should be noted that the value of c for Fig. 1(a) is different than that used by Lakhtakia et al. for $\alpha = 5$. Based on these and other images, we state the first conjecture using the following notation:

- λ_s^z - a self-similar major lobe in the z-plane fractal image.
- Λ^z - the total number of self-similar major lobes in the z-plane fractal image.
- $\Omega^z(\lambda_s^z)$ - the angular space spanned by λ_s^z around the origin.

The superscript z is used to differentiate from the notation used for the c-plane fractal images in [GUJA90].

Conjecture 1. $\mathcal{J}^z$ for $\alpha = \eta$ *contains* Λ^z *self-similar major lobes given by*

$$\Lambda^z = \eta\lambda_s^z \textit{ with } \Omega^z(\lambda_s^z) \cong 2\pi/\eta.$$

4. Positive Non-Integer α

When α is not an integer, i.e. $\alpha = +(\eta + \varepsilon)$, the fractal images generated from the even and odd values of η, as ε changes from 0 to 1, evolve differently, even though the underlying mathematical formulation and the image generating algorithm are the same. These cases are discussed below.

The fractal image for $\alpha = 3.3$ is given in Fig. 2(a). We see that there are three self-similar major lobes and a small embryonic lobe on the left-hand side. When α is increased to 3.6, we obtain Fig. 2(b) where the embryonic lobe has increased in size. The embryonic lobe is always above the real axis on the negative side. Further, there is a clear line demarking the embryonic lobe and the major lobe below it. This discontinuity arises due to the fact that the arguments of the complex quantities have to be restricted to some interval of length 2π and this is usually chosen as $-\pi$ to $+\pi$ or 0 to 2π. We have verified that when $\alpha = 4$, this embryonic lobe develops into a new major lobe giving rise to four self-similar major lobes. Several fractal images have been generated for other odd values of η as ε changes from 0 to 1.

We propose Conjecture 2 using the following additional notation:

- λ_e^z - an embryonic lobe.
- $\Omega^z(\lambda_e^z)$ - the angular space spanned by λ_e^z around the origin.

Conjecture 2. *$\mathcal{J}^z$ for $\alpha = +(\eta + \varepsilon)$, with η as an odd integer,consists of Λ^z self-similar major lobes given by $\Lambda^z = \eta\lambda_s^z$ with $\Omega(\lambda_s^z) \cong 2\pi/(\eta + \varepsilon)$ and an embryonic lobe λ_e^z with $\Omega(\lambda_e^z) \cong 2\pi\varepsilon/(\eta + \varepsilon)$; the embryonic lobe is always above and on the negative side of the real axis.*

The fractal image for $\alpha = 4.5$ is given in Fig. 2(c). There are four self-similar major lobes and an embryonic lobe on the left hand side in between the two similar major lobes. When the value of α is increased to 4.8, this embryonic lobe grows (see Fig. 2(d)). The growth of the embryonic lobe is proportional to the value of ε. This embryonic lobe develops into a major lobe when $\alpha = 5$ (see Fig. 1(a)). It has been observed that growth of this embryonic lobe is always below and on the negative side of the real axis. Based on our experiments, we state Conjecture 3.

Conjecture 3. *$\mathcal{J}^z$ for $\alpha = +(\eta + \varepsilon)$ and η as an even integer consists of Λ^z self-similar major lobes given by $\Lambda^z = \eta\lambda_s^z$ with $\Omega(\lambda_s^z) \cong 2\pi/(\eta + \varepsilon)$ and an embryonic lobe λ_e^z with $\Omega(\lambda_e^z) \cong 2\pi\varepsilon/(\eta + \varepsilon)$; the embryonic lobe is always below and on the negative side of the real axis.*

5. Negative Integer α

The fractal image for $\alpha = -4$ (see Fig. 3(a)) resembles a planetary structure with a central planet surrounded by satellite structures. There are four major satellite structures. Figure 3(b) shows the image for $\alpha = -5$ where one can see five major satellite structures. Thus, it is seen that the number of major satellite structures is always equal to the value of η. The satellite structures are situated symmetrically around the origin. The stable region (represented by the white color) surrounds the unstable region; this is just the reverse of what happens for the positive values of α where the unstable region surrounds the stable region.

We propose Conjecture 4 using the following additional notation:

- Ξ^z - the total number of major satellite structures.
- ξ_s^z - a major satellite structure.
- $\Omega(\xi_s^z)$ - the angular space spanned by ξ_s^z around the origin.

Conjecture 4. *$\mathcal{J}^z$ for $\alpha = -\eta$ consists of Ξ^z major satellite structures given by $\Xi^z = \eta\xi_s^z$ with $\Omega(\xi_s^z) \cong 2\pi/\eta$.*

6. Negative Non-Integer α

The fractal images for odd and even values of η evolve somewhat differently and therefore are discussed separately in this section.

The fractal image generated for $\alpha = -3.3$ is given in Fig 4(a). There are three major satellite structures and an embryonic satellite structure on the left hand side displacing all the major satellite structures. As the value of α decreases to -3.6 the embryonic satellite grows in size (see Fig. 4(b)). The embryonic satellite structure emerges below the real axis and there is a definite line between the embryonic satellite structure and the other major satellites. This can again be attributed to the restriction on the arguments of the complex quantities. We state Conjecture 5 using the following additional notation:

- ξ_e^z - an embryonic satellite structure.
- $\Omega(\xi_e^z)$ - the angular space spanned by ξ_e^z around the origin.

Conjecture 5. *$\mathcal{J}^z$ for $\alpha = -(\eta + \varepsilon)$ and η as an odd integer consists of Ξ^z major satellite structures, given by $\Xi^z = \eta\xi_s^z$ with $\Omega(\xi_s^z) \cong 2\pi/(\eta + \varepsilon)$ and an embryonic satellite structure ξ_e^z with $\Omega(\xi_e^z) \cong 2\pi\varepsilon/(\eta + \varepsilon)$; the embryonic satellite structure is always below and on the negative side of the real axis.*

Figures 4(c) and 4(d) contain the fractal images for α = -4.6 and α = -4.9, respectively. There are four major satellite structures and the space between the two left-most major satellite structures has increased displacing all the major satellite structures proportionally. The new embryonic satellite structure emerges above and on the negative side of the real axis. When α = -5, the number of major satellite structures increases to five (see Fig. 3(b)). This shows that the fractal image grows smoothly from $\alpha = -(\eta + \varepsilon)$ to $\alpha = -(\eta + 1)$. Based on these and other observations we formulate Conjecture 6 as follows.

Conjecture 6. *$\mathcal{J}^z$ for $\alpha = -(\eta + \varepsilon)$ and η as an even integer consists of Ξ^z major satellite structures given by $\Xi^z = \eta\xi_s^z$ with $\Omega(\xi_s^z) \cong 2\pi/(\eta + \varepsilon)$ and an embryonic satellite structure ξ_e^z with $\Omega(\xi_e^z) \cong 2\pi\varepsilon/(\eta + \varepsilon)$; the embryonic satellite structure is always above and on the negative side of the real axis.*

7. Conclusion

Several fractal images generated from the transformation function $z \leftarrow z^{\alpha} + c$ in the complex z-plane have been presented. The images for positive integer values of α exhibit lobular structures. The number of major lobes is equal to α. When α is a negative integer number, the image exhibits a panetory configuration consisting of a central planet surrounded by several major satellite structures. The number of major satellite structures is equal to $|\alpha|$. For non-integer values of α an additional embryonic lobular/satellite structure, whose size is proportional to the fractional part of α, emerges. We have proposed six conjectures about the visual characteristics and the symmetries in these fractal images.

Acknowledgements

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Figures

The color photographs of the fractal images have been submitted to the editor. This copy of the paper contains only the reduced, black and white photocopies of these images.

Figure Captions

Fig 1. Fractal images for postive integer α.

(a) $\alpha = 5$
(b) $\alpha = 12$

Fig 2. Fractal images for positive non-integer α.

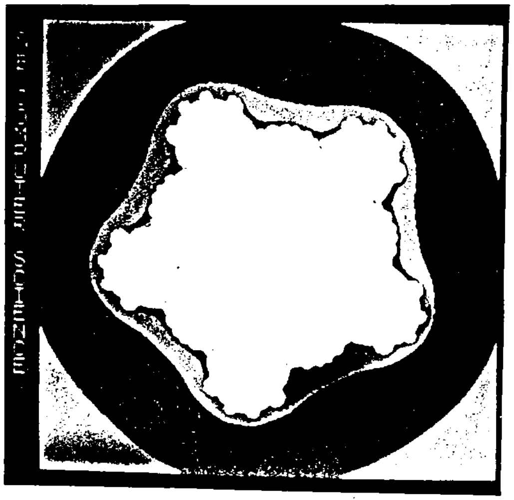
(a) $\alpha = 3.3$ (c) $\alpha = 4.5$
(b) $\alpha = 3.6$ (d) $\alpha = 4.8$

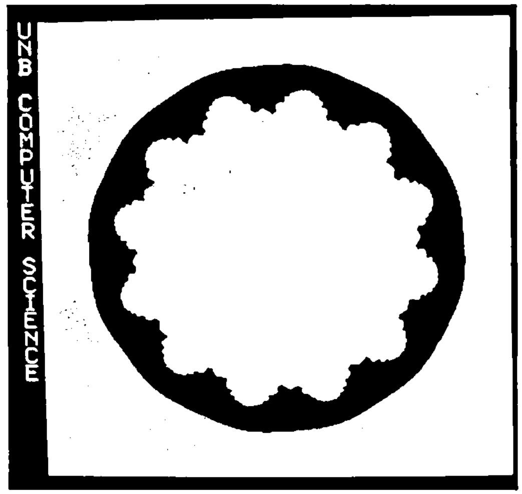
Fig 3. Fractal images for negative integer α.

(a) $\alpha = -4$
(b) $\alpha = -5$

Fig 4. Fractal images for negative non-integer α.

(a) $\alpha = -3.3$ (c) $\alpha = -4.6$
(b) $\alpha = -3.6$ (d) $\alpha = -4.9$

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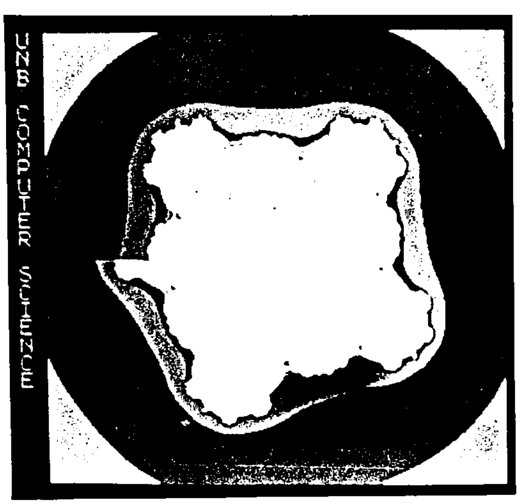
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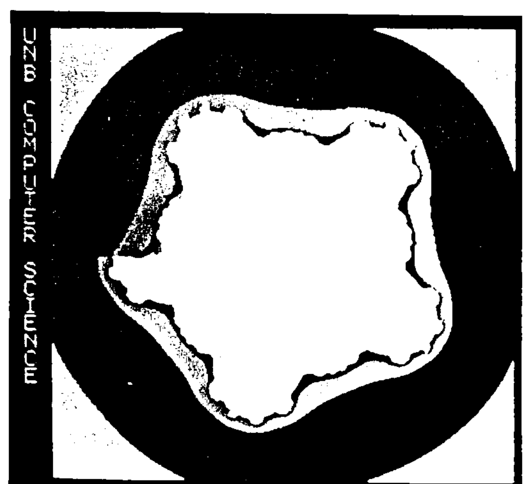
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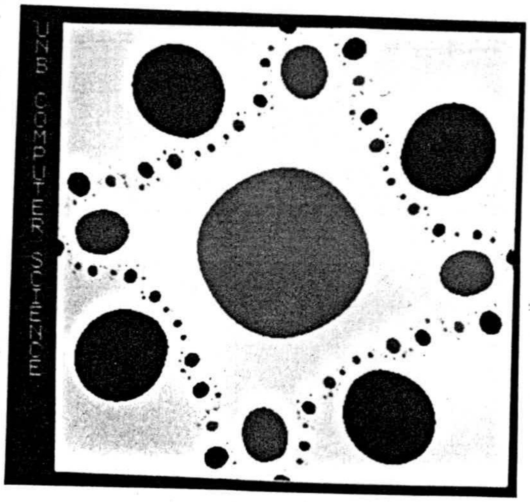
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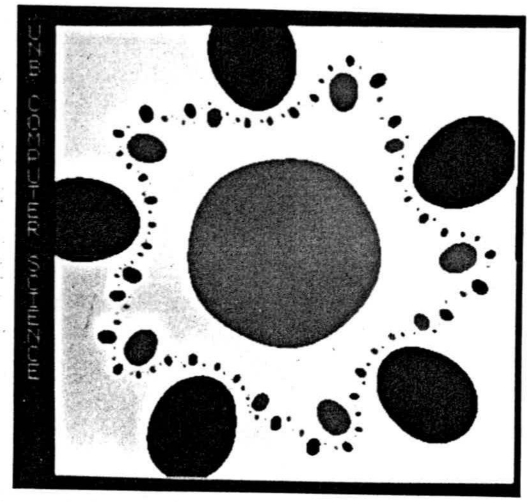
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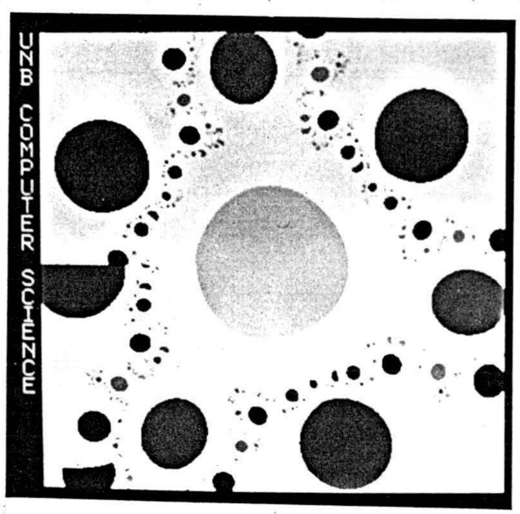
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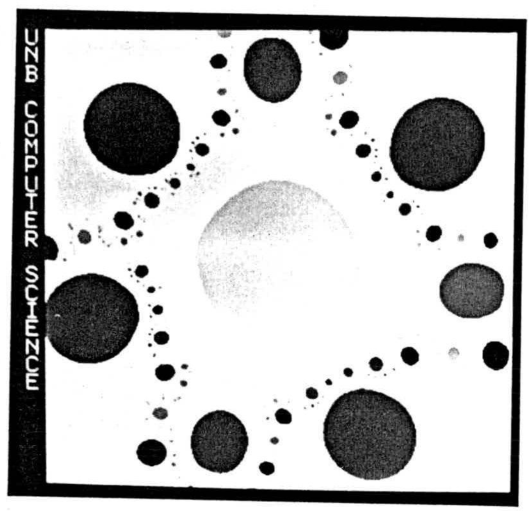
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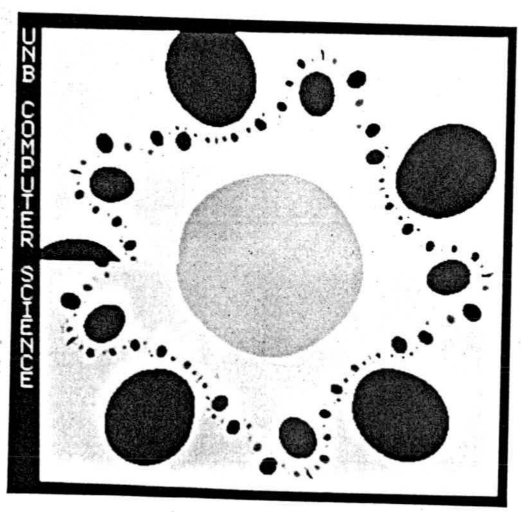
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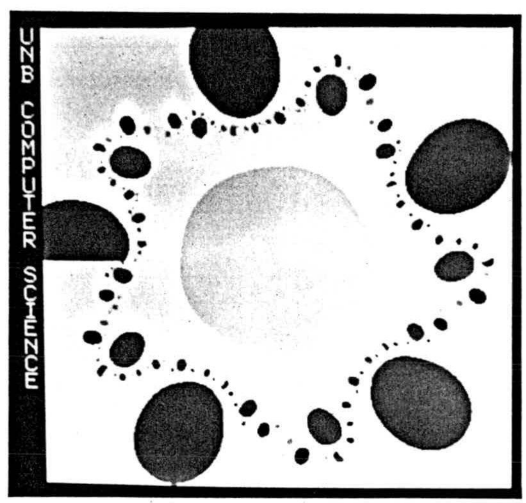
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