Lee and Manhattan MWS and FWS Codes

by

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Abstract

Let q be a prime number and let $V = \mathbb{F}_q^n$ be the vector space consisting of all the length n vectors whose components are elements of the finite field $\mathbb{F}_q$. We say that $C \subseteq \mathbb{F}_q^n$ is a linear code if C is a subspace of V, namely for every two elements $\overline{\mathbf{c}}_\mathbf{1}, \overline{\mathbf{c}}_\mathbf{2} \in C$ and two scalars $a_1, a_2 \in \mathbb{F}_q$ we have $a_1 \cdot \overline{\mathbf{c}}_\mathbf{1} + a_2 \cdot \overline{\mathbf{c}}_\mathbf{2} \in C$. The elements of C are called codewords. The finite field $\mathbb{F}_q$ which a code is over is called the alphabet. The space $\mathbb{F}_q^n$ may be endowed with a weight function w, which can be induced by a distance metric. Hamming weight, the weight function induced by the Hamming Metric, is the most widely used in coding theory. The size q of the alphabet, and the dimension k of a linear code impose a maximum number of Hamming weights that a linear code can have, denoted $L_H(k, q)$. In 2018, Dr. T. Alderson and Dr. A. Neri released a publication titled "Maximum Weight Spectrum Codes" in which it was shown for any prime power q and any positive integer k that $L_H(k, q) = \frac{q^k-1}{q-1}$. Linear codes in which there are $\frac{q^k-1}{q-1}$ distinct Hamming weights were given the name maximum weight spectrum (MWS) codes. In this work we determine that for any prime number q and any positive integer k the maximum number of distinct Lee weights that a linear code can have is $L_L(k, q) = \frac{q^k-1}{2}$ and the maximum number of distinct Manhattan weights that a linear code can have is $L_M(k, q) = q^k - 1$. This is done by first identifying a theoretical upper bound on the functions $L_L(k, q)$ and $L_M(k, q)$, and then showing the bound is sharp by constructing codes which meet the bound with equality. We show that with respect to Hamming, Lee, and

Manhattan weights, there is a lower bound on the length of MWS codes of $n = \frac{q^k-1}{q-1}$. Linear codes in which there is at least one codeword of each weight observed in $\mathbb{F}_q^n$ are called full weight spectrum (FWS) codes. In 2018, Alderson authored an article titled "A note on Full Weight Spectrum Codes" in which it was shown that the maximum length of a Hamming FWS code is $n = 2^k - 1$. We show that with respect to Lee and Manhattan weights, there is an upper bound on the length of FWS codes. Namely, we show that the maximum length of a Lee FWS code is at least $n = \frac{(\frac{q+1}{2})^k-1}{(\frac{q+1}{2})^k-1}$ and the maximum length of a Manhattan FWS code is $n = \frac{q^k-1}{q-1}$. We leave a generalized result for the maximum length of an FWS code with respect to a componentwise metric as an open problem.

Dedication

I would like to dedicate this work to my beautiful daughter, Dilan Audrey, who came into my life and made it better. I also dedicate this work to my mother, Beverly Morine, who was lost to cancer in 2017 and will never stop being missed.

> Be kind, tenderhearted, forgiving one another.
>
> Ephesians 4:32

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List of Symbols, Nomenclature or Abbreviations

Unless otherwise noted, this list of symbols are defined as follows:

C	A set of vectors which make up a code.
$\overline{\mathbf{c}}$	An element(codeword) of C.
n	The length of a code.
k	The dimension of a code.
q	A prime power.
$d(C)$(Or just d)	The minimum distance between any two codewords of C.
G	A generator matrix for C.
$\mathbb{F}_q$	The finite field with q elements.
$\mathbb{F}_q^n$	The n-dimensional vector space over $\mathbb{F}_q$.
$[n,k]_q$-code	A linear code with length n, dimension k, over $\mathbb{F}_q$.
$[n,k,d]_q$-code	A linear code with length n, dimension k, minimum distance d, over $\mathbb{F}_q$.
$w(\overline{\mathbf{c}})$	The weight of a codeword.
$w(C)$	The weight set of a code.
$L(k,q)$	The maximum possible number of weights in a linear code given k and q.
$L(k,q,n)$	The maximum possible number of weights in a linear code given k,q, and n.

Chapter 1

Introduction

A *linear code* C of *length* n and *dimension* k is a k-dimensional subspace of the vector space $V = \mathbb{F}_q^n$ where $\mathbb{F}_q$ is the finite field with q elements. We shall refer to V as the *ambient space* of C. The code C is called a q-ary code. If $q = 2$ or $q = 3$ then the code is a *binary* code or a *ternary* code respectively. The vectors in C are called *codewords*. Note that necessarily we have the *size* of a code (the number of codewords) is q^k. Codewords represent information that can be transmitted between a sender and a receiver. A key purpose of a code is to allow communication across channels in such a way that information lost in transmission can be recovered. The ambient space V may be equipped with a *metric* which is a function that gives the distance $d(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ between pairs of vectors in V as a non negative real number. Metrics used in coding theory are typically integer valued functions [7]. The distance from a vector $\overline{\mathbf{x}}$ to the origin $d(\overline{\mathbf{x}}, \mathbf{0})$ is called the *metric induced weight* of $\overline{\mathbf{x}}$, denoted $w_d(\overline{\mathbf{x}})$. The Hamming distance between two vectors $d_H(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ is the number of coordinates in which they differ. The Hamming Weight of a vector $w_H(\overline{\mathbf{x}})$ is the number of non-zero coordinates. The number of distinct Hamming distances in a code is an important parameter with respect to the reliability with which information lost in transmission can be recovered. In particular, the minimum distance between codewords is of importance [8]. In the case of linear codes, the minimum Hamming distance is the Hamming weight of the codeword of least weight [1]. The weight set of a code $w(C)$ is the set of distinct non-zero weights observed in the code. In 2019, Shi et al. released an article titled "How many weights can a linear code have", in which the authors pointed out an upper bound on the size of the Hamming-weight set of a linear code. The bound given can be easily proved but its tightness is nontrivial. However, the bound was later shown to be tight for all dimensions k and any prime power q [1]. The maximum number of distinct weights $L(k, q)$ that a k-dimesnional q-ary code can have has been studied with respect to the Hamming Metric [1]. A refinement of the $L(k, q)$ function which considers codes of fixed length n, $L(n, k, q)$, has also been studied with respect to the Hamming Metric. It is known for any positive integer k and any prime power q that $L_H(k, q) = \frac{q^k-1}{q-1}$ [1]. Codes in which $|w_H(C)| = \frac{q^k-1}{q-1}$ were given the name Maximum Weight Spectrum (MWS) codes [1]. If the weight set of a linear code is equal in size to that of the ambient space V, meaning there is at least one codeword in C of each possible weight, we say that code is a Full Weight Spectrum (FWS) code. In [2], it was shown that Hamming FWS codes exist if and only if $n < 2^k$. Although the Hamming metric is by far the dominant metric used in coding theory, there are applications in which the Lee and Manhattan metrics are more favorable [5, 7]. Both MWS and FWS codes are a quite new area of study and so far work in the literature has only been done in the Hamming metric. The intent of this work is to define MWS and FWS codes with respect to Lee and Manhattan weight, and to examine under what conditions they exist. For Lee-weight and Manhattan-weight, an upper bound is established on the size of the weight set of a linear code and the bound is shown to be tight by constructing matrices which generate codes that meet the bound with equality. A tight upper bound on the length of a Manhattan FWS code as a function of k and q is determined whereas the same for a Lee FWS code has been conjectured but not completely determined.

To help understand the main problem tackled in this work, which is the number of Lee and Manhattan weights that a linear code can have, we will begin by defining finite fields, vector spaces, metrics, and weight functions.

1.1 Fields

Definition: A *field* $\mathbb{F}$ is a set of elements equipped with two operations + (called addition) and $\cdot$ (called multiplication) in which the following properties are satisfied for every element a, b, c in $\mathbb{F}$:

i. Closure: $(a + b) \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$

ii. Commutativity: $a + b = b + a$ and $a \cdot b = b \cdot a$

iii. Associativity: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

iv. Distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$

v. There is an additive identity element $0 \in \mathbb{F}$ such that $a + 0 = a$

vi. There is a multiplicative identity element $1 \in \mathbb{F}$ such that $a \cdot 1 = a$

vii. There is an additive inverse element $-a \in \mathbb{F}$ such that $a + (-a) = 0$

viii. If $a \neq 0$, there is a multiplicative inverse element $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$

Note: a set R which is not necessarily commutative under multiplication but otherwise satisfying properties (i) to (vii) is called a *Ring*.

The above definitions for *field* and *ring* can be found in [10].

1.1.1 Prime Fields

A field can have an infinite number of elements, for example $\mathbb{Q}$ or $\mathbb{R}$, or a finite number of elements.

Definition: A *finite field* $\mathbb{F}_q$ is a field with a finite number of elements. The number of elements q in a finite field is called the *order* of the field.

French mathematician Evariste Galois (1811-1832) proved that finite fields of order q exist if and only if $q = p^n$ for some prime number p and some $n \in \mathbb{N}$. Finite fields of order q are unique up to relabeling [8].

Definition: a *prime field*, denoted $\mathbb{F}_p$, is a finite field with prime order.

The elements of a prime field are the set of integer residues modulo p for some prime number p.

$$\mathbb{F}_p = \{0, 1, \ldots, p-1\} \tag{1.1}$$

Addition, subtraction, and multiplication of elements in $\mathbb{F}_p$ are performed using the usual operations on the integers, followed by reduction modulo p. To reduce an integer modulo p means to first divide by p, and then keep the remainder as the result. For example, 19 reduced modulo 7 is 5, since dividing 19 by 7 yields a remainder of 5. We write $17 \equiv 5 \mod 7$. Division is multiplication by the inverse modulo p, followed by reduction modulo p. Since p is prime it follows that for every $a \in \mathbb{F}_p \setminus \{0\}$, $\gcd(a, p) = 1$. By the extended Euclidian Algorithm we can obtain integers s and t such that

$$a \cdot s + p \cdot t = \gcd(a, p) = 1$$

Reading the above equation reduced modulo p we see that

$$a \cdot s \equiv 1 \mod p.$$

So $s \mod p$ is the inverse of a modulo p. Thus each element of $\mathbb{F}_p \setminus \{0\}$ has a multiplicative inverse (field axiom($viii$)). Another consequence of the primality of p is that there are no two elements $a, b \in \mathbb{F}_p \setminus \{0\}$ such that the product $a \cdot b = 0$. We say that $\mathbb{F}_p$ has no zero divisors. Therefore when working over $\mathbb{F}_p$, any product equal to 0 must have 0 as a factor. Moreover, the lack of zero divisors in a field gives us the *cancellation property*. If for some $a, b, c \in \mathbb{F}_p \setminus \{0\}$ we have that $a \cdot b = a \cdot c$, then we may conclude that $a \cdot b - a \cdot c = 0 \implies a \cdot (b - c) = 0 \implies b = c$ [10].

1.2 Vector Spaces

Definition: A *vector space* consists of a set V of elements called vectors, and a field $\mathbb{F}$ of elements called scalars, equipped with two operations:

- Vector addition. If $\overline{\mathbf{u}}, \overline{\mathbf{v}} \in V$ then $\overline{\mathbf{u}} + \overline{\mathbf{v}} \in V$
- Scalar multiplication. If $a \in \mathbb{F}$ and $\overline{\mathbf{u}} \in V$ then $a \cdot \overline{\mathbf{u}} \in V$

The following properties are satisfied for every $\overline{\mathbf{u}}, \overline{\mathbf{v}}, \overline{\mathbf{w}} \in V$ and every $a, b \in \mathbb{F}$:

i. Associativity of vector addition: $(\overline{\mathbf{u}} + \overline{\mathbf{v}}) + \overline{\mathbf{w}} = \overline{\mathbf{u}} + (\overline{\mathbf{v}} + \overline{\mathbf{w}})$

ii. Commutativity of vector addition: $\overline{\mathbf{u}} + \overline{\mathbf{v}} = \overline{\mathbf{v}} + \overline{\mathbf{u}}$

iii. There is an unique all zero vector $\mathbf{0} \in V$ such that $\overline{\mathbf{u}} + \mathbf{0} = \overline{\mathbf{u}}$

iv. There is a negative vector $-\overline{\mathbf{u}} \in V$ such that $\overline{\mathbf{u}} + (-\overline{\mathbf{u}}) = \mathbf{0}$

v. Associativity of scalar multiplication: $(a \cdot b)\overline{\mathbf{u}} = a \cdot (b \cdot \overline{\mathbf{u}})$

vi. Distributivity: $(a + b) \cdot \overline{\mathbf{u}} = a \cdot \overline{\mathbf{u}} + b \cdot \overline{\mathbf{u}}$ and $a \cdot (\overline{\mathbf{u}} + \overline{\mathbf{v}}) = a \cdot \overline{\mathbf{u}} + a \cdot \overline{\mathbf{v}}$

vii. $1 \cdot \overline{\mathbf{u}} = \overline{\mathbf{u}}$ where 1 is the multiplicative identity in $\mathbb{F}$.

Notes:

1. We say that V is a *vector space over* $\mathbb{F}$.

2. If a nonempty set $U \subseteq V$ is itself a vector space over $\mathbb{F}$, we call U a *subspace* of V.

3. Let $S = \{\overline{\mathbf{v}}_\mathbf{1}, \overline{\mathbf{v}}_\mathbf{2}, \ldots, \overline{\mathbf{v}}_\mathbf{k}\} \subseteq V$:

   i. We call the set $\langle S \rangle = \{a_1 \cdot \overline{\mathbf{v}}_\mathbf{1} + a_2 \cdot \overline{\mathbf{v}}_\mathbf{2} + \ldots a_k \cdot \overline{\mathbf{v}}_\mathbf{k} \mid a_1, a_2, \ldots, a_k \in \mathbb{F}\}$ the *span* of S. We may also denote the span of a set of vectors explicitly as $\langle \overline{\mathbf{v}}_\mathbf{1}, \overline{\mathbf{v}}_\mathbf{2}, \ldots, \overline{\mathbf{v}}_\mathbf{n} \rangle$.

   ii. For any $a_1, a_2, \ldots, a_k \in \mathbb{F}$ we call $a_1 \cdot \overline{\mathbf{v}}_\mathbf{1} + a_2 \cdot \overline{\mathbf{v}}_\mathbf{2} + \ldots a_k \cdot \overline{\mathbf{v}}_\mathbf{k}$ a *linear combination* of the vectors in S. If $a_1 \cdot \overline{\mathbf{v}}_\mathbf{1} + a_2 \cdot \overline{\mathbf{v}}_\mathbf{2} + \ldots a_k \cdot \overline{\mathbf{v}}_\mathbf{k} = \mathbf{0}$ implies that $a_1 = a_2 = \cdots = a_k = 0$ we say that the set S is *linearly independent.*

4. The span of any subset of V is a subspace of V.

5. If a set $B \subseteq V$ is linearly independent then we say that B is a *basis* for $\langle B \rangle$. The number of vectors in B is the *dimension* of $\langle B \rangle$.

6. Every vector space has a basis.

The vector spaces of study in this work will be the n-dimensional vector space over a prime field of order q, denoted $\mathbb{F}_q^n$.

$$\mathbb{F}_q^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{F}_q\} \tag{1.2}$$

1.2.1 Number of Subspaces of a Finite Vector Space

The Gaussian binomial coefficient $\binom{n}{m}_q$ counts, among other things, the number of m-dimensional subspaces of the vector space $V = \mathbb{F}_q^n$ [9]. It is given by

$$\binom{n}{m}_q = \frac{(q^n - 1)(q^n - q) \ldots (q^n - q^{m-1})}{(q^m - 1)(q^m - q) \ldots (q^m - q^{m-1})} \tag{1.3}$$

1.3 Metrics and Weight Functions

Definition: A *metric* d on a set X is a function

$$d : X \times X \to [0, \infty)$$

that satisfies the following properties for every $x, y, z \in X$:

i. $d(x, y) = 0 \iff x = y$

ii. $d(x, y) = d(y, x)$ (Symmetry)

iii. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality)

Notes:

1. We say that $d(x, y)$ is the *distance* from x to y.

2. The pair (X, d) is called a *metric space*.

There are various ways to define a weight function. For our purposes we shall adopt the following.

Definition: A *Weight function* w on the vector space $\mathbb{F}_q^n$ is a function satisfying the following two properties:

1. $w : \mathbb{F}_q^n \to \mathbb{Z}^+ \cup \{0\}$

2. $w(\overline{\mathbf{x}}) = 0 \iff \overline{\mathbf{x}} = \mathbf{0}$

Definition: The *Weight function* w_d *induced by the metric* d is the distance from $\overline{\mathbf{x}}$ to the all zero vector.

$$w_d(\overline{\mathbf{x}}) = d(\overline{\mathbf{x}}, \mathbf{0})$$

In a metric induced weight function, the distance between two elements in a vector space $d(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ is the weight of their difference $w_d(\overline{\mathbf{x}} - \overline{\mathbf{y}})$ [7]. By metric property (ii), symmetry, we have that $w_d(\overline{\mathbf{x}}) = d(\overline{\mathbf{x}}, \overline{\mathbf{0}}) = d(\overline{\mathbf{0}}, \overline{\mathbf{x}}) = w_d(\overline{\mathbf{0}} - \overline{\mathbf{x}}) = w_d(-\overline{\mathbf{x}})$.

Lemma 1.3.1. *If w_d is a metric induced weight function on $\mathbb{F}_q^n$ then,*

$$w_d(\overline{\mathbf{x}}) = w_d(-\overline{\mathbf{x}}) \textit{ for all } \overline{\mathbf{x}} \in \mathbb{F}_q^n$$

Chapter 2

Preliminaries

We will use $\mathbb{F}_q$ to represent the finite field with q elements. When working in the Lee Metric, we will assume throughout that $\mathbb{F}_q$ is a prime field. $\mathbb{F}_q^n$ is the n-dimensional vector space over $\mathbb{F}_q$. For an element $\overline{\mathbf{x}} \in \mathbb{F}_q^n$ we write $\overline{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$ where each x_i is called the i'th coordinate of x. For a codeword $\overline{\mathbf{c}} \in C$, we write $\overline{\mathbf{c}} = c_1c_2\ldots c_n$ with the coordinates concatenated. If $q \geq 10$, an alphabetic symbol is used to denote multiple digit coordinates. A k-dimensional q-ary code of length n will be denoted succinctly as an $[n, k]_q$ code.

2.1 Weight Functions to be Examined

2.1.1 Hamming Weight

The *Hamming Weight* of a scalar $a \in \mathbb{F}_q$ is defined as follows:

$$w_H(a) = \begin{cases} 0, & a = 0 \\ 1, & a \neq 0 \end{cases} \tag{2.1}$$

The *Hamming Weight* of a vector $\overline{\mathbf{x}} \in \mathbb{F}_q^n$ is:

$$w_H(\overline{\mathbf{x}}) = \sum_{i=1}^{n} w_H(x_i) \tag{2.2}$$

2.1.2 Lee Weight

The *Lee Weight* of a scalar $a \in \mathbb{Z}_m$ is:

$$w_L(a) = \min\{a, m - a\} \tag{2.3}$$

The *Lee Weight* of a vector $\overline{\mathbf{x}} \in \mathbb{Z}_m^n$ is:

$$w_L(\overline{\mathbf{x}}) = \sum_{i=1}^{n} w_L(x_i) \tag{2.4}$$

2.1.3 Manhattan Weight

The *Manhattan Weight* of an integer $a \in \mathbb{Z}$ is:

$$w_M(a) = |a| \tag{2.5}$$

The *Manhattan Weight* of an element $\overline{\mathbf{x}} \in \mathbb{Z}^n$ is:

$$w_M(\overline{\mathbf{x}}) = \sum_{i=1}^{n} w_M(x_i) \tag{2.6}$$

Note that the Manhattan Weight function is a metric induced weight function on $\mathbb{Z}^n$.

2.2 Definitions

Definition: Let w be a weight function on $\mathbb{F}_q^n$. For $X \subseteq \mathbb{F}_q^n$, the *weight set* of X, denoted $w(X)$, is the set of nonzero weights realized by the elements of X:

$$w(X) = \{w(\overline{\mathbf{x}}) \mid \overline{\mathbf{x}} \in X \setminus \{\mathbf{0}\}\} \tag{2.7}$$

Definition: The *sphere* of radius r and center $\overline{\mathbf{x}}_0$, denoted $S_d(\overline{\mathbf{x}}_0, r)$ is the set of points in $\mathbb{F}_q^n$ which are at a distance r from $\overline{\mathbf{x}}_0$ with respect to metric d. Note that for the metrics discussed in this thesis the cardinality of a sphere is independent of $\overline{\mathbf{x}}_0$.

$$S_d(\overline{\mathbf{x}}_0, r) = \{\overline{\mathbf{x}} \in \mathbb{F}_q^n \mid d(\overline{\mathbf{x}}, \overline{\mathbf{x}}_0) = r\} \tag{2.8}$$

Definition: A *generator matrix* G for a linear code C is a $k \times n$ matrix over $\mathbb{F}_q$ in which the rows of G are a basis for C.

Definition: A weight function on an n-dimensional vector space V is said to be *component-wise* if the weight of a vector $\overline{\mathbf{x}} \in V$ is given by $w_d(\overline{\mathbf{x}}) = \sum_{i=1}^{n} w_d(x_i)$.

Definition: The *support* of a vector $\overline{\mathbf{v}} \in \mathbb{F}_q^n$, denoted $\text{supp}(\overline{\mathbf{v}})$, is the set of indices of nonzero coordinates of $\overline{\mathbf{v}}$:

$$\text{supp}(\overline{\mathbf{v}}) = \{i \mid v_i \neq 0\} \tag{2.9}$$

Definition: We use $n_{d,MWS}(k, q)$ to denote the minimum length of an MWS code with respect to weight function d:

$$n_{d,MWS}(k, q) = \min\{n \in \mathbb{N} \mid \exists \text{ an } [n, k]_q \text{ d-MWS code}\} \tag{2.10}$$

2.3 Maximum Weight Functions, MWS and FWS Codes

We use $L_d(k,q)$ to denote the maximum number of distinct nonzero w_d-weights that a k-dimensional q-ary code can have. We use $L_d(n,k,q)$ to denote the maximum number of distinct nonzero w_d-weights that a k-dimensional q-ary code of length n can have. These functions are the main objects of study with respect to MWS codes.

2.3.1 Maximum Weight Spectrum Codes

Definition: A *Maximum Weight Spectrum* (MWS) *Code* C is an $[n,k]_q$ code in which:

$$|w_d(C)| = L_d(k,q) \tag{2.11}$$

2.3.2 Full Weight Spectrum Codes

Definition: A *Full Weight Spectrum* (FWS)*Code* C is an $[n,k]_q$ code in which:

$$|w_d(C)| = |w_d(\mathbb{F}_q^n)| \tag{2.12}$$

Chapter 3

Bounds on Cardinality of Weight Set

Proposition 3.0.1. *Let w be a component-wise weight function on the vector space $V = \mathbb{F}_q^n$.*

1. *If $|w(\langle \overline{\mathbf{u}} \rangle)| \leq D$ for every $\overline{\mathbf{u}} \in V$ then $L(k,q) \leq \frac{D(q^k-1)}{q-1}$.*

2. *If $\max\limits_{a \in \mathbb{F}_q} w(a) = m$ then $|w(V)| \leq n \cdot m$ with equality if $w(\mathbb{F}_q) = \{1, \ldots, m\}$.*

Proof.

1. Let C be an $[n,k]_q$ code. Recall that equation (1.3) counts m-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$. Since C is a k-dimensional space, putting $n = k$ and $m = 1$ gives us that C has exactly $\binom{k}{1}_q = \frac{q^k-1}{q-1}$ 1-dimensional subspaces. Pick $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2, \ldots, \overline{\mathbf{c}}_{(\frac{q^k-1}{q-1})} \in C$ such that $C = \langle \overline{\mathbf{c}}_1 \rangle \cup \langle \overline{\mathbf{c}}_2 \rangle \cup$

$\cdots \cup \langle \overline{\mathbf{c}}_{(\frac{q^k-1}{q-1})} \rangle$. We have that:

$$
\begin{aligned}
w(C) &= w(\langle \overline{\mathbf{c}}_1 \rangle \cup \langle \overline{\mathbf{c}}_2 \rangle \cup \cdots \cup \langle \overline{\mathbf{c}}_{(\frac{q^k-1}{q-1})} \rangle) \\
\implies |w(C)| &= |w(\langle \overline{\mathbf{c}}_1 \rangle \cup \langle \overline{\mathbf{c}}_2 \rangle \cup \cdots \cup \langle \overline{\mathbf{c}}_{(\frac{q^k-1}{q-1})} \rangle)| \\
&\leq |w(\langle \overline{\mathbf{c}}_1 \rangle)| + |w(\langle \overline{\mathbf{c}}_2 \rangle)| + \cdots + |w(\langle \overline{\mathbf{c}}_{(\frac{q^k-1}{q-1})} \rangle)| \\
&\leq \underbrace{D + \cdots + D}_{\frac{q^k-1}{q-1} \text{times}} \\
&= \frac{D(q^k-1)}{q-1}
\end{aligned}
$$

2. Let $\overline{\mathbf{v}}$ be a vector in V. Since $\overline{\mathbf{v}}$ has n coordinates, the weight of each coordinate is at most m, and w is a component-wise weight function, we have that $0 \leq w(\overline{\mathbf{v}}) \leq n \cdot m$. So $w(V) \subseteq \{1, \ldots, n \cdot m\}$ which implies that $|w(V)| \leq n \cdot m$. Now suppose that $w(\mathbb{F}_q) = \{1, \ldots, m\}$. Then the map from V to $\{1, \ldots, n \cdot m\}$ given by $\mathbf{x} \mapsto w(\mathbf{x})$ is surjective. So $w(V) = \{1, \ldots, n \cdot m\}$ which implies that $|w(V)| = n \cdot m$.

□

Corollary 3.0.1. *Let w be a component-wise weight function and let D and m be as defined above. If C is an $[n, k]_q$-code then we have the following:*

1. $|w(C)| = \frac{D(q^k-1)}{q-1} \implies n \geq \frac{D(q^k-1)}{m(q-1)}$
2. $|w(C)| = m \cdot n \implies n \leq \frac{D(q^k-1)}{m(q-1)}$

Proof.

1. Suppose C is an $[n, k]_q$-code with $|w(C)| = \frac{D(q^k-1)}{q-1}$. Clearly the number of

distinct nonzero weights in C is not greater than that of V. So,

$$\begin{aligned} |w(C)| &\leq |w(V)| \\ \Longrightarrow \quad \tfrac{D(q^k-1)}{q-1} &\leq n \cdot m \\ \Longrightarrow \quad n &\geq \tfrac{D(q^k-1)}{m(q-1)} \end{aligned}$$

2. Suppose C is an $[n,k]_q$-code with $|w(C)| = n \cdot m$. By definition, the number of distinct nonzero weights in C is less than or equal to $L(k,q)$. So,

$$\begin{aligned} |w(C)| &\leq L(k,q) \\ \Longrightarrow \quad n \cdot m &\leq \tfrac{D(q^k-1)}{q-1} \\ \Longrightarrow \quad n &\leq \tfrac{D(q^k-1)}{m(q-1)} \end{aligned}$$

$\square$

Chapter 4

Maximum Weight Spectra

The general bound on $L(k, q)$ for a componentwise weight function given part 1 of Proposition 3.0.1 gives us an easy upper bound on the size of the Lee and Manhattan weight set of a linear code. What is not trivial is the sharpness of such bound, i.e. do there exist codes whose weight set meets the bound with equality or is there a tighter bound to be established? To answer help this question, an exhaustive search method has been taken. Starting with small fixed values of k and q and incrementally increasing n we generate every linear code with those parameters and check the size of each weight set. We stop once we have found that $L(n, k, q)$ achieves the upper bound on $L(k, q)$ from part 1 of Proposition 3.0.1 or when the number of codes to check has become too extensive based on computational capacity. Results are shown in tables throughout chapter 4. As we will see by the end of Chapter 4, the bound on the size of the Lee and Manhattan weight set of linear code, as in Proposition 3.0.1, turns out to be tight for all positive integers k and all prime numbers q ($q > 2$ in the case of Lee weight).

4.1 MWS Codes in the Hamming Metric

We will use $w_H(\overline{\mathbf{x}})$ to denote the Hamming Weight of a vector $\overline{\mathbf{x}} \in \mathbb{F}_q^n$. It is easy to verify that nonzero multiples of a vector in $\mathbb{F}_q^n$ share the same Hamming weight. That is, for every $\overline{\mathbf{x}} \in \mathbb{F}_q^n$ and every $a \in \mathbb{F}_q \setminus \{0\}$, $w_H(\overline{\mathbf{x}}) = w_H(a\overline{\mathbf{x}})$. In particular, for every $\overline{\mathbf{x}} \in \mathbb{F}_q^n$, $|w_H(\langle \overline{\mathbf{x}} \rangle)| \leq 1$. Thus, in the context of Proposition 3.0.1, the Hamming weight on $\mathbb{F}_q^n$ satisfies $D = m = 1$ giving that for any linear code C, $|w_H(C)| \leq \frac{q^k-1}{q-1}$ and if $|w_H(C)| = \frac{q^k-1}{q-1}$ then $n \geq \frac{q^k-1}{q-1}$. Another proof of this result is given in [11]. In [1] it was shown that for all positive integers k and any prime power q, $L_H(k,q) = \frac{q^k-1}{q-1}$. This was shown by constructing very long $[n,k]_q$-codes whose weight set has cardinality $\frac{q^k-1}{q-1}$. To address some gaps in the bounds regarding "short" codes I used computational software to exhaustively check the Hamming weight sets of the span of every set of k linearly independent vectors from $\mathbb{F}_q^n$. The following tables show $L_H(n,k,q)$ for the indicated values of k and q. Note that for $k = 1$ all nonzero codewords are nonzero multiples of each other, whence $L(n,1,q) = 1$.

Table 4.1: Results for 2-dimensional Binary Codes in The Hamming Metric

$q = 2$	$\frac{q^k-1}{q-1} = 3$		
Length	**Number of Codes**	$\mathbf{L_H(n,2,2)}$	**Notes**
$n = 2$	1	2	
$n = 3$	7	3	$L_H(2,2) = 3$ $n_{H,MWS}(2,2) = 3$ 3 MWS codes
$n = 4$	35	3	16 MWS codes
$n = 5$	155	3	65 MWS codes
$n = 6$	651	3	291 MWS codes

An example of a generator matrix for a length 3 MWS binary code is

$$G = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An example of a generator matrix for a length 6 MWS binary code is

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Table 4.2: Results for 2-dimensional Ternary Codes in The Hamming Metric

$q = 3$	$\frac{q^k-1}{q-1} = 4$		
Length	**Number of Codes**	$\mathbf{L_H(n, 2, 3)}$	**Notes**
$n = 2$	1	2	
$n = 3$	13	3	
$n = 4$	130	3	
$n = 5$	1210	3	
$n = 6$	11011	4	$L_H(2, 3) = 4$ $n_{H,MWS}(2, 3) = 6$ 1920 MWS codes
$n = 7$	99463	4	$20,160$ MWS codes
$n = 8$	896260	n/a	Exhaustive search too large

An example of a generator matrix for a length 6 MWS ternary code is

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

An example of a generator matrix for a length 7 MWS ternary code is

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$

Table 4.3: Results for 2-dimensional 5-ary Codes in The Hamming Metric

$q = 5$	$\frac{q^k-1}{q-1} = 6$		
Length	**Number of Codes**	$\mathbf{L_H(n, 2, 5)}$	**Notes**
$n = 2$	1	2	
$n = 3$	31	3	
$n = 4$	806	3	
$n = 5$	20306	3	
$n = 6$	508431	4	$n_{H,MWS}(2, 5) > 6$
$n = 7$	12714681	n/a	Exhaustive search too large

4.2 MWS Codes in the Lee Metric

Note that for $q \leq 3$ the Lee Metric is equivalent to the Hamming Metric [5]. Therefore we will begin investigating MWS codes in the Lee Metric with $q = 5$. Because Lee-weight is a metric induced weight function, by Lemma 1.3.1 from Chapter 1 we have that for any vector $\overline{\mathbf{x}} \in \mathbb{F}_q^n$, $w_L(\overline{\mathbf{x}}) = w_L(-\overline{\mathbf{x}})$. Because of this, we get the following corollary:

Corollary 4.2.1. *If C is an $[n, k]_q$-code with $q > 2$, then for any $t > 0$*

$$|S_L(0, t) \cap C| \in 2\mathbb{Z}.$$

Proof. If $S_L(0,t) \cap C = \emptyset$ then $|S_L(0,t) \cap C| = 0 \in 2\mathbb{Z}$. If $S_L(0,t) \cap C \neq \emptyset$ then for every $\overline{\mathbf{x}} \in S_L(0,t) \cap C$ we have that $-\overline{\mathbf{x}} \in S_L(0,t) \cap C$. Note that since $q > 2$, $\overline{\mathbf{x}}$ and $-\overline{\mathbf{x}}$ are distinct elements. $\square$

Because every $\overline{\mathbf{x}} \in \mathbb{F}_q^n$ has $w_L(\overline{\mathbf{x}}) = w_L(-\overline{\mathbf{x}})$, we have that $w_L(2\overline{\mathbf{x}}) = w_L(-2\overline{\mathbf{x}})$, and so on. In particular, $|w_L(\langle \overline{\mathbf{x}} \rangle)| \leq \lfloor \frac{q}{2} \rfloor = \frac{q-1}{2}$ (since q is assumed odd). Note that $\max\limits_{a \in \mathbb{F}_q} w_L(a) = \frac{q-1}{2}$ since if $a > \lfloor \frac{a}{2} \rfloor$, $w_L(a) = q - a$. Thus, in the context of Proposition 3.0.1, the Lee Metric on $\mathbb{F}_q^n$ satisfies $D = m = \frac{q-1}{2}$ giving the following corollary:

Corollary 4.2.2. *If C is an $[n,k]_q$-code with $q > 2$ prime, then*

1. $|w_L(C)| \leq \frac{q^k-1}{2}$.
2. $|w_L(C)| = \frac{q^k-1}{2} \implies n \geq \frac{q^k-1}{q-1}$.

Lemma 4.2.1. *If C is an $[n,k]_q$-code with $q > 2$ and with $|w_L(C)| = \frac{q^k-1}{2}$, then for any $t > 0$,*

$$|S_L(0,t) \cap C| = 0 \text{ or } |S_L(0,t) \cap C| = 2$$

Proof. For each $t \in w_L(C)$, $|S_L(0,t)| \geq 2$ by Corollary 2. If any such sphere had cardinality greater than 2 then the number of nonzero codewords would exceed $2|w_L(C)|$ giving $q^k - 1 > q^k - 1$, a contradiction. $\square$

Lemma 4.2.2. *If C is an $[n,k]_q$ code with $q > 2$ and with $|w_L(C)| = \frac{q^k-1}{2}$ then for any $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2 \in C$ we have that*

$$w_L(\overline{\mathbf{c}}_1) = w_L(\overline{\mathbf{c}}_2) \iff \overline{\mathbf{c}}_1 = \pm\overline{\mathbf{c}}_2.$$

Proof. Follows from Lemma 1.3.1 and from Lemma 4.2.1. $\square$

Lemma 4.2.3. *Let C be an $[n,k]_q$-code with $q > 2$ and suppose that there exist nonzero $\overline{\mathbf{c}}_1 \neq \overline{\mathbf{c}}_2 \in C$ with $\operatorname{supp}(\overline{\mathbf{c}}_1) \cap \operatorname{supp}(\overline{\mathbf{c}}_2) = \emptyset$. Then*

$$|w_L(C)| < \frac{q^k - 1}{2}.$$

Proof. Suppose for some nonzero $\overline{\mathbf{c}}_1 \neq \overline{\mathbf{c}}_2 \in C$ we have that $\text{supp}(\overline{\mathbf{c}}_1) \cap \text{supp}(\overline{\mathbf{c}}_2) = \emptyset$. Then $w_L(\overline{\mathbf{c}}_1 + \overline{\mathbf{c}}_2) = w_L(\overline{\mathbf{c}}_1) + w_L(\overline{\mathbf{c}}_2) = w_L(\pm\overline{\mathbf{c}}_1) + w_L(\pm\overline{\mathbf{c}}_2) = w_L(\pm\overline{\mathbf{c}}_1 \pm \overline{\mathbf{c}}_2)$. Note that since $q > 2$, $\overline{\mathbf{c}}_1 \neq \mathbf{0} \neq \overline{\mathbf{c}}_2$ and $\overline{\mathbf{c}}_1 \neq \overline{\mathbf{c}}_2$ we have that $(\overline{\mathbf{c}}_1 + \overline{\mathbf{c}}_2)$, $(\overline{\mathbf{c}}_1 - \overline{\mathbf{c}}_2)$, and $(\overline{\mathbf{c}}_2 - \overline{\mathbf{c}}_1)$ are mutually distinct codewords. So $|S_L(0,t)| \cap C > 2$ and the result follows from Lemma 4.2.1. □

Corollary 4.2.3. *If C is an $[n,k]_q$-code with $q > 2$ and with $|w_L(C)| = \frac{q^k-1}{2}$ and B is a basis for C, then for any $\overline{\mathbf{u}}, \overline{\mathbf{v}} \in B$,*

$$\text{supp}(\overline{\mathbf{u}}) \cap \text{supp}(\overline{\mathbf{v}}) \neq \emptyset$$

The following tables show $L_L(n,k,q)$ for the indicated values of k and q. The values in tables $4.4 - 4.5$ were found by use of computational software to exhaustively check the Lee weight sets of the span of every set of k linearly independent vectors from $\mathbb{F}_q^n$. Note that since when $q \in \{2,3\}$ the Lee Metric is equivalent to the Hamming Metric, our results for the Lee Metric begin with $q = 5$.

Table 4.4: Results for 1-dimensional 5-ary Codes in The Lee Metric

$q = 5$	$\frac{q^k-1}{2} = 2$	$\frac{q^k-1}{q-1} = 1$	
Length	**Number of Codes**	$\mathbf{L_L(n,1,5)}$	**Notes**
$n = 1$	1	2	$L_L(1,5) = 2$ $n_{1,5} = 2$
$n = 2$	6	2	4 MWS codes

The four 1-dimensional, length 2, MWS 5-ary codes have the following generator matrices:

$$G_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, G_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \text{ and } G_4 = \begin{bmatrix} 1 & 4 \end{bmatrix}$$

Table 4.5: Results for 2-dimensional 5-ary Codes in The Lee Metric

$q = 5$	$\frac{q^k-1}{2} = 12$	$\frac{q^k-1}{q-1} = 6$	
Length	**Number of Codes**	$\mathbf{L_L(n, 2, 5)}$	**Notes**
$n = 2$	1	4	
$n = 3$	31	6	
$n = 4$	806	8	
$n = 5$	$20,306$	8	
$n = 6$	$508,431$	9	
$n = 7$	$12,714,681$	9	$L_L(2, 5) \geq 9$ $n_{L,MWS}2, 5 \geq 6$

We notice a pattern that when $2 < n \leq 6$ is even, $L_L(n, 2, 5) = L_L(n + 1, 2, 5)$

Proposition 4.2.1. *There exists an* $[11, 2]_5$ *code* C *with* $|W_L(C)| = 12$

Proof. Let G be the following 2×11 matrix over $\mathbb{F}_5$:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The $[11, 2]_5$ code C generated by G has 24 nonzero codewords shown here with Lee Weight indicated:

Weight 1	Weight 2	Weight 3	Weight 4	Weight 5
none	none	none	40000000111	none
			10000000444	

Weight 6	Weight 7	Weight 8	Weight 9	Weight 10
none	none	30000000222	21111111000	none
		20000000333	34444444000	

Weight 11	Weight 12	Weight 13	Weight 14	Weight 15
11111111111	24444444111	01111111222	14444444222	42222222000
44444444444	31111111444	04444444333	41111111333	13333333000

Weight 16	Weight 17	Weight 18	Weight 19	Weight 20
none	03333333111	none	32222222111	none
	02222222444		23333333444	

Weight 21	Weight 22
43333333222	22222222222
12222222333	33333333333

□

Corollary 4.2.4. $L_L(2,5) = 12$ *and* $8 \leq n_{L,MWS}(2,5) \leq 11$

We will now generalize the generator matrix in Proposition 4.2.2 by fixing the first row of the given generator matrix at all 1's and letting the second row be combinations of 0's, 1's, and -1's. Working over $\mathbb{F}_5$ lets consider the two dimensional code $C = \langle \overline{\mathbf{u}}, \overline{\mathbf{v}} \rangle$ where $\overline{\mathbf{u}}$ is the all 1 vector and the coordinates of $\overline{\mathbf{v}}$ consist of a -1's, b 1's, and zeros such that $a+b < n$ and $0 < a < b$. In order for C to be an MWS code the 12 nonzero codewords in C shown in the first two columns of the following table must have distinct Lee weight:

$\overline{\mathbf{c}}$	$2\overline{\mathbf{c}}$	$w_L(\overline{\mathbf{c}})$	$w_L(2\overline{\mathbf{c}})$
$\overline{\mathbf{u}}$	$2\overline{\mathbf{u}}$	n	$2n$
$\overline{\mathbf{v}}$	$2\overline{\mathbf{v}}$	$a+b$	$2a+2b$
$\overline{\mathbf{u}}+\overline{\mathbf{v}}$	$2\overline{\mathbf{u}}+2\overline{\mathbf{v}}$	$n-a+b$	$2n-2a-b$
$\overline{\mathbf{u}}+2\overline{\mathbf{v}}$	$2\overline{\mathbf{u}}+4\overline{\mathbf{v}}$	$n+b$	$2n-b$
$2\overline{\mathbf{u}}+\overline{\mathbf{v}}$	$4\overline{\mathbf{u}}+2\overline{\mathbf{v}}$	$2n-a$	$n+a$
$\overline{\mathbf{u}}+4\overline{\mathbf{v}}$	$2\overline{\mathbf{u}}+3\overline{\mathbf{v}}$	$n+a-b$	$2n-a-2b$

The weights shown above are distinct if and only if the following conditions are satisfied:

$$n \neq 2(a+b) \tag{4.1}$$

$$n \neq a+2b, b+2a \tag{4.2}$$

$$n \neq a+3\frac{b}{2}, b+3\frac{a}{2} \tag{4.3}$$

$$n \neq a+3b, b+3a \tag{4.4}$$

$$n \neq 2a+3\frac{b}{2}, 2b+3\frac{a}{2} \tag{4.5}$$

$$n \neq 2b-a, 2a-b \tag{4.6}$$

$$n \neq 2a, 2b \tag{4.7}$$

$$n \neq 3a, 3b \tag{4.8}$$

$$b \neq 2a \tag{4.9}$$

The following shows that MWS codes of this type do not exist with length $n = 8, 9, 10$:

$\underline{n=8}$

(a,b)	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(2,3)$	$(2,4)$	$(2,5)$	$(3,4)$
Condition violated	4.9	4.1	4.7	4.4	4.2	4.2	4.2	4.3	4.7

$\underline{n = 9}$

(a, b)	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(1, 6)$	$(1, 7)$	$(2, 3)$	$(2, 5)$	$(2, 6)$
Condition violated	4.9	4.8	4.2	4.6	4.4	4.2	4.4	4.2	4.3

(a, b)	$(3, 4)$	$(3, 5)$
Condition violated	4.3	4.8

$\underline{n = 10}$

(a, b)	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(1, 6)$	$(1, 7)$	$(1, 8)$	$(2, 3)$	$(2, 4)$
Condition violated	4.9	4.8	4.1	4.7	4.3	4.4	4.2	4.1	4.2

(a, b)	$(2, 5)$	$(2, 6)$	$(2, 7)$	$(3, 4)$	$(3, 5)$	$(3, 6)$	$(4, 5)$
Condition violated	4.7	4.2	4.3	4.2	4.7	4.9	4.7

The generator matrix given in Proposition 4.2.2 corresponds to $n = 11$ with $(a, b) = (1, 3)$. One can verify that conditions $4.1 - 4.9$ are satisfied for $n = 11$ if $(a, b) \in \{(1, 3), (1, 4), (1, 7), (4, 6)\}$.

The next theorem and proposition, Theorem 4.2.1 and Proposition 4.2.2, are given without proof. For proofs see [3]. Theorem 4.2.1 and Proposition 4.2.2 will be useful in proving Proposition 4.2.3 and Proposition 4.2.4 in which we will show that $L_L(k, q) = \frac{q^k - 1}{2}$ for any positive integer k and any prime number $q > 2$.

Theorem 4.2.1 (Basis Representation Theorem)**.** *Let b be any integer larger than 1. Then, for each positive integer n, there exists a representation*

$$n = a_0 + a_1 b + a_2 b^2 + \cdots + a_s b^s$$

where $a_s \neq 0$, and where each a_i is non negative and less than b. Furthermore, this representation of n is unique, it is called the representation of n to the base b.

Proposition 4.2.2. *If*

$$a_0 + ab + ... + a_s b^s$$

is a representation of n to the base b, then n is any positive integer less than or equal to $b^{s+1} - 1$.

Lemma 4.2.4. *Let* $a, b \in \mathbb{Z}_q$ *with* $q > 2$ *prime. Then*

$$w_L(a-b) = w_L(a+b) \iff a = 0 \text{ or } b = 0.$$

Proof.

$$\begin{aligned} w_L(a-b) = w_L(a+b) &\iff a-b = a+b \text{ or } a-b = -(a+b) \\ &\iff 2b = 0 \text{ or } 2a = 0 \\ &\iff b = 0 \text{ or } a = 0 \text{ (since } q > 2 \text{ prime)} \end{aligned}$$

□

Proposition 4.2.3. *Let G be the following matrix over* $\mathbb{F}_q$ *with* $q > 2$ *prime:*

$$G = \begin{bmatrix} 1 & \overbrace{1 \dots 1}^{(\frac{q+1}{2})} & \overbrace{0 \dots 0}^{(\frac{q+1}{2})^2} \\ 1 & 0 \dots 0 & 1 \dots 1 \end{bmatrix}$$

The code generated by G is an MWS code.

In the following proof we use the notation: $|m| = w_L(m)$ in $\mathbb{Z}_q$

Proof. Let C be the code generated by G. We will show that C is an MWS code by showing that if two codewords $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2 \in C$ have the same Lee weight then $\overline{\mathbf{c}}_1 = \pm\overline{\mathbf{c}}_2$. Let $\overline{\mathbf{c}}_1 = (a, b) \cdot G$ and $\overline{\mathbf{c}}_2 = (s, t) \cdot G$ for some $a, b, s, t \in \mathbb{F}_q$. Let $\alpha = (q+1)/2$. Then

$w_L(\overline{\mathbf{c}}_1) = |a+b| + |a| \cdot \alpha + |b| \cdot \alpha^2$ and $w_L(\overline{\mathbf{c}}_2) = |s+t| + |s| \cdot \alpha + |t| \cdot \alpha^2$. Therefore,

$$w_L(\overline{\mathbf{c}}_1) = w_L(\overline{\mathbf{c}}_2) \iff |a+b| + |a| \cdot \alpha + |b| \cdot \alpha^2 = |s+t| + |s| \cdot \alpha + |t| \cdot \alpha^2$$

Since $|a+b|,\ |a|,\ |s+t|,\ |s| \in \{0, 1, \ldots, \frac{q-1}{2} = \alpha - 1\}$, both sides of the equation above is an integer uniquely represented in base α. Thus, we can equate coefficients to get the following:

i. $|a+b| = |s+t|$

ii. $|a| = |s|$

iii. $|b| = |t|$

$(ii.)$ implies that $s = \pm a$ and $(iii.)$ implies that $t = \pm b$ which together give the following four cases:

Case 1: $\mathbf{s = a}$ and $\mathbf{t = -b}$ Putting $s = a$ and $t = -b$ into $(i.)$ gives $|a+b| = |a-b|$ which by Lemma 4.2.4 implies that $a = 0$ or $b = 0$. Assume $a = 0$. Then $s = 0$ so $\overline{\mathbf{c}}_1 = (0, b) \cdot G$ and $\overline{\mathbf{c}}_2 = (0, \pm b) \cdot G$ thus $\overline{\mathbf{c}}_1 = \pm \overline{\mathbf{c}}_2$. Now assume $a \neq 0$ and $b = 0$. Then $\overline{\mathbf{c}}_1 = (a, 0) \cdot G$ and $\overline{\mathbf{c}}_2 = (a, 0) \cdot G$ so $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_{\mathbf{2}}$. Finally, if $a = b = 0$ then $s = t = 0$ so $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$.

Case 2: $\mathbf{s = -a}$ and $\mathbf{t = b}$ By a similar argument to that made in Case 1, $\overline{\mathbf{c}}_1 = \pm \overline{\mathbf{c}}_2$.

Case 3: $\mathbf{s = -a}$ and $\mathbf{t = -b}$ In this case $(a, b) = (-s, -t)$ so $\overline{\mathbf{c}}_1 = -\overline{\mathbf{c}}_2$.

Case 4: $\mathbf{s = a}$ and $\mathbf{t = b}$ In this case $(a, b) = (s, t)$ so $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$.

Thus we have in each case either $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$ or $\overline{\mathbf{c}}_1 = -\overline{\mathbf{c}}_2$ which concludes the proof.

□

The following shows existence of MWS codes of shorter length than in Proposition 4.2.3. Codes with $C = \langle \overline{\mathbf{u}}, \overline{\mathbf{v}} \rangle$ where

$$\overline{\mathbf{u}} = 2 \quad \overbrace{1 \ldots 1}^{(q+1)/2 \text{ times}} \quad \overbrace{0 \ldots 0}^{\text{x times}}$$

$$\overline{\mathbf{v}} = -1 \quad \overbrace{0 \ldots 0}^{(q+1)/2 \text{ times}} \quad \overbrace{1 \ldots 1}^{x \text{ times}}$$

with x chosen according to Table 4.6

Table 4.6: "Shorter" length of 2-dimensional Lee MWS codes

q	5	7	11	13	17	19	23
x	7	11	27	38	66	75	113

were verified using computational software to have $|w_L(C)| = \frac{q^k-1}{2}$. Therefore we have following bounds on minimum length:

Table 4.7: Bounds on Length of 2-dimensional MWS Codes in the Lee Metric

$\mathbf{q}$	$\mathbf{n_{L,MWS}(2, q)}$
5	$8 \leq n_{\text{L,MWS}}(2, 5) \leq 11$
7	$8 \leq n_{\text{L,MWS}}(2, 7) \leq 16$
11	$12 \leq n_{\text{L,MWS}}(2, 11) \leq 34$
13	$14 \leq n_{\text{L,MWS}}(2, 13) \leq 46$
17	$18 \leq n_{\text{L,MWS}}(2, 17) \leq 76$
19	$20 \leq n_{\text{L,MWS}}(2, 19) \leq 86$
23	$24 \leq n_{\text{L,MWS}}(2, 23) \leq 126$

The next proposition demonstrates that $L_L(k, q) = \frac{q^k-1}{2}$ for any dimension k and any prime $q > 2$.

Proposition 4.2.4. *Let k be a positive integer and let $\overline{\mathbf{e}}_i$ be the length k column vector with the ith coordinate 1 and 0s elsewhere. Let $q > 2$ be prime and let G be the following $k \times n$ matrix over $\mathbb{F}_q$ where $\alpha = (q+1)/2$.*

$$G = \begin{bmatrix} \overline{\mathbf{e}}_1 & | \overbrace{\overline{\mathbf{e}}_2 \dots \overline{\mathbf{e}}_2}^{\alpha \ times} | & \dots & | \overbrace{\overline{\mathbf{e}}_k \dots \overline{\mathbf{e}}_k}^{\alpha^{k-1} \ times} | & \overbrace{(\overline{\mathbf{e}}_1 + \overline{\mathbf{e}}_2) \dots (\overline{\mathbf{e}}_1 + \overline{\mathbf{e}}_2)}^{\alpha^k \ times} | & \dots & | & \overbrace{(\overline{\mathbf{e}}_i + \overline{\mathbf{e}}_j) \dots (\overline{\mathbf{e}}_i + \overline{\mathbf{e}}_j)}^{i \neq j \ \forall \ i,j \in \{1,\dots,k\}, \alpha^{k-1+\binom{k}{2}} \ times} & | \end{bmatrix}$$

The linear code generated by G is a Lee MWS code.

In the following proof we use the notation: $|m| = w_L(m)$ in $\mathbb{Z}_q$

Proof. Let C be the linear code generated by G. We will show that C is a MWS code by showing that if any two codewords $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2 \in C$ share the same Lee Weight then $\overline{\mathbf{c}}_1 = \pm\overline{\mathbf{c}}_2$. By definition $\overline{\mathbf{c}}_1 = (a_1, \dots, a_k) \cdot G$ for some $a_1, \dots, a_k \in \mathbb{F}_q$ and $\overline{\mathbf{c}}_2 = (b_1, \dots, b_k) \cdot G$ for some $b_1, \dots, b_k \in \mathbb{F}_q$. Assume $w_L(\overline{\mathbf{c}}_1) = w_L(\overline{\mathbf{c}}_2)$. Then,

$$|a_1| + |a_2| \cdot \alpha + \dots + |a_k| \cdot \alpha^{k-1} + |a_1 + a_2| \cdot \alpha^k \dots = |b_1| + |b_2| \cdot \alpha + \dots + |b_k| \cdot \alpha^{k-1} + |b_1 + b_2| \cdot \alpha^k \dots$$

Since each coefficient above is an element of $\{0, 1, \dots, (q-1)/2 = \alpha - 1\}$, the left and right hand sides of the above equation is an integer uniquely represented in base α. We can therefore compare coefficients giving:

i. $|a_i| = |b_i|$ for $i = 1, \dots, k$.

ii. $|a_i + a_j| = |b_i + b_j|$ for $i \neq j \in \{1, \dots, k\}$.

(i.) implies that for every $i \in \{1, \dots, k\}$ either $a_i = b_i$ or $a_i = -b_i$ from which we get the following cases:

Case 1: $\mathbf{a_i = b_i}$ for $\mathbf{i = 1, \dots, k}$. In this case, $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$

Case 2: $\mathbf{a_i = -b_i}$ for $\mathbf{i = 1, \dots, k}$. In this case, $\overline{\mathbf{c}}_1 = -\overline{\mathbf{c}}_2$

Case 3: for some $\mathbf{I} \subset \{\mathbf{1}, \ldots, \mathbf{k}\}, \mathbf{a_i} = \mathbf{b_i}$ for $\mathbf{i} \in \mathbf{I}$ and $\mathbf{a_j} = -\mathbf{b_j}$ for $\mathbf{j} \in \{\mathbf{1}, \ldots, \mathbf{k}\} \setminus \mathbf{I}$

In this case, for $j \in \{1, \ldots, k\} \setminus I$, putting $a_j = -b_j$ into (*ii.*) gives $|a_i + a_j| = |a_i - a_j|$ which, since $q > 2$, by Lemma 4.2.2 implies that $a_i = 0$ or $a_j = 0$. If $a_i = 0$ for every $i \in I$ then $b_i = 0$ for every $i \in I$ and since $a_j = -b_j$ for every $j \in \{1, \ldots, k\} \setminus I$, we have that $\overline{\mathbf{c}}_1 = -\overline{\mathbf{c}}_2$. On the other hand, if $a_j = 0$ for every $j \in \{1, \ldots, k\} \setminus I$ then $b_j = 0$ for every $j \in \{1, \ldots, k\} \setminus I$ and since $a_i = b_i$ for every $i \in I$, we have that $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$.

Thus in each case either $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$ or $\overline{\mathbf{c}}_1 = -\overline{\mathbf{c}}_2$ which concludes the proof. □

Proposition 4.2.3 shows that for all positive integers k and for all $q > 2$ prime,

$$L_L(k, q) = \frac{q^k - 1}{2}.$$

Corollary 4.2.5. *TFAE:*

1. *C is a Lee MWS code.*
2. *$|w_L(C)| = \frac{q^k-1}{2}, q > 2$.*
3. *$w_L(\overline{\mathbf{c}}_1) = w_L(\overline{\mathbf{c}}_2) \iff \overline{\mathbf{c}}_1 = \pm\overline{\mathbf{c}}_2$, for every $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2 \in C$.*

4.3 Manhattan MWS Codes

We will use $w_M(x)$ to denote the Manhattan weight of a vector $x \in \mathbb{F}_q^n$. In the next lemma we establish sufficiency conditions for the set of scalar multiples of a vector to have distinct Manhattan weights.

Lemma 4.3.1. *Let $\overline{\mathbf{x}}$ be a vector in $\mathbb{F}_q^n$ with q prime and let α and β be nonzero scalars in $\mathbb{F}_q$.*

$$w_M(\alpha\overline{\mathbf{x}}) = w_M(\beta\overline{\mathbf{x}}) \implies \alpha = \beta \text{ or } \sum_{i=1}^{n} x_i \equiv 0 \mod q$$

Proof. Note that for $\alpha, \beta \neq 0$:

$$\begin{aligned}
w_M(\alpha\overline{\mathbf{x}}) = w_M(\beta\overline{\mathbf{x}}) &\implies \sum_{i=1}^{n} \alpha \cdot x_i = \sum_{i=1}^{n} \beta \cdot x_i \\
&\implies \sum_{i=1}^{n} \alpha \cdot x_i \equiv \sum_{i=1}^{n} \beta \cdot x_i \qquad \bmod q \\
&\implies \alpha \cdot \sum_{i=1}^{n} x_i \equiv \beta \cdot \sum_{i=1}^{n} x_i \qquad \bmod q \\
&\implies (\alpha - \beta) \cdot \sum_{i=1}^{n} x_i \equiv 0 \quad \bmod q \\
&\implies \alpha = \beta \text{ or } \sum_{i=1}^{n} x_i \equiv 0 \quad \bmod q \text{ (since } q \text{ is prime)}
\end{aligned}$$

□

Considering Lemma 4.3.1, it may be possible to have a linear code in which each codeword is of distinct Manhattan Weight. In other words, this would mean that $|w_M(C)| = q^k - 1$. The following tables show $L_M(n, k, q)$ for the indicated values of k and q. The values in tables $4.7 - 4.12$ were found by use of computational software to exhaustively check the Manhattan weight sets of the span of every set of k linearly independent vectors from $\mathbb{F}_q^n$.

Table 4.8: Results for 1-dimensional 2-ary Manhattan Weights

$q = 2$	$q^k - 1 = 1$	$\frac{q^k-1}{q-1} = 1$	
Length	**Number of Codes**	$\mathbf{L_M(n, 1, 2)}$	**Notes**
$n = 1$	1	1	$L_M(1, 2) = 1$
			$n_{M,MWS}(1, 2) = 1$
			1 MWS code
$n = 2$	3	1	3 MWS codes

Table 4.9: Results for 2-dimensional 2-ary Manhattan Weights

$q = 2$	$q^k - 1 = 3$	$\frac{q^k-1}{q-1} = 3$	
Length	**Number of Codes**	$\mathbf{L_M(n, 2, 2)}$	**Notes**
$n = 2$	1	2	
$n = 3$	7	3	$L_M(2, 2) = 3$
			$n_{M,MWS}(2, 2) = 3$
			3 MWS codes

Table 4.10: Results for 1-dimensional 3-ary Manhattan Weights

$q = 3$	$q^k - 1 = 2$	$\frac{q^k-1}{q-1} = 1$	
Length	**Number of Codes**	$\mathbf{L_M(n, 1, 3)}$	**Notes**
$n = 1$	1	2	$L_M(1, 3) = 2$
			$n_{L,MWS}(1, 3) = 1$
			1 MWS code
$n = 2$	4	2	3 MWS codes

Table 4.11: Results for 2-dimensional 3-ary Manhattan Weights

$q=3$	$q^k-1=8$	$\frac{q^k-1}{q-1}=4$	
Length	**Number of Codes**	$\mathbf{L_M(n,2,3)}$	**Notes**
$n=2$	1	4	
$n=3$	13	6	
$n=4$	130	8	$L_M(2,3)=8$ $n_{L,MWS}(2,3)=4$ 4 MWS codes
$n=5$	1210	8	35 MWS codes
$n=6$	$11,011$	8	276 MWS codes

Table 4.12: Results for 1-dimensional 5-ary Manhattan Weights

$q=5$	$q^k-1=4$	$\frac{q^k-1}{q-1}=1$	
Length	**Number of Codes**	$\mathbf{L_M(n,1,5)}$	**Notes**
$n=1$	1	4	$L_M(1,5)=4$ $n_{M,MWS}(1,5)=1$ 1 MWS code
$n=2$	6	4	5 MWS codes

Table 4.13: Results for 2-dimensional 5-ary Manhattan Weights

$q = 5$	$q^k - 1 = 24$	$\frac{q^k-1}{q-1} = 6$	
Length	**Number of Codes**	$\mathbf{L_M(n,1,5)}$	**Notes**
$n = 2$	1	8	
$n = 3$	31	12	
$n = 4$	806	16	
$n = 5$	20306	20	
$n = 6$	508431	24	$L_M(2,5) = 24$ $n_{M,MWS}(1,5) = 6$

The next proposition shows that indeed $L_M(k,q) = q^k - 1$ for any prime q and any dimension k.

Proposition 4.3.1. *Let k be a positive integer and let $\overline{\mathbf{e}}_i$ be the length k column vector with the i'th coordinate 1 and 0s elsewhere. Let q be prime and and let G be following matrix over $\mathbb{F}_q$:*

$$G = \begin{bmatrix} \overline{\mathbf{e}}_1 & | \overbrace{\overline{\mathbf{e}}_2 \ldots \overline{\mathbf{e}}_2}^{q \text{ times}} | & \ldots & | \overbrace{\overline{\mathbf{e}}_k \ldots \overline{\mathbf{e}}_k}^{q^{k-1} \text{ times}} | \end{bmatrix}$$

Then G generates a Manhattan MWS code.

Proof. Let C be the code generated by G. We will show that C is a Manhattan MWS code by showing that if two codewords $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2 \in C$ share the same Manhattan Weight then $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$. By definition $\overline{\mathbf{c}}_1 = (a_1, \ldots, a_k) \cdot G$ for some $a_1, \ldots, a_k \in \mathbb{F}_q$ and $\overline{\mathbf{c}}_2 = (b_1, \ldots, b_k) \cdot G$ for some $b_1, \ldots, b_k \in \mathbb{F}_q$. Assume $w_M(\overline{\mathbf{c}}_1) = w_M(\overline{\mathbf{c}}_2)$. Use the notation $w_M(a) = |a|$ for $a \in \mathbb{F}_q$.

$$|a_1| + |a_2| \cdot q + \ldots |a_k| q^{k-1} = |b_1| + |b_2| \cdot q + \ldots |b_k| q^{k-1}$$

Since the coefficients above are elements of $\{0, 1, \ldots, q-1\}$, the left and right hand side of the equation above are integers uniquely represented in base q. Therefore we can equate coefficients to get that $|a_i| = |b_i|$ for every $i \in \{1, \ldots, k\}$. Since two scalars in $\mathbb{F}_q$ have the same Manhattan weight only when they are equal, this implies that $a_i = b_i$ for every $i \in \{1, \ldots, k\}$, therefore $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$. □

The above generator matrix has $n = 1+q+\cdots+q^{k-1} = \frac{q^k-1}{q-1}$ so the code C generated by G has $|w_M(C)| = q^k - 1 = n(q-1)$, therefore C is also a Manhattan FWS code.

Chapter 5

Full Weight Spectra

Part 2 of Proposition 3.0.1 can be used to give us the size of the Lee and Manhattan weight set of the space $V = \mathbb{F}_q^n$. This can be used as a check as to whether a linear code is FWS with respect to each weight function (C is FWS if $|w(C)| = |w(V)|$). Starting with small fixed values of k and q and incrementally increasing n we generate every linear code with those parameters and check the size of each weight set with the goal of discovering for which values of k and q there is a value of n such that $[n, k]_q$ FWS codes exist. We stop when either we have found that there exists an FWS code or when n reaches the upper bound of an FWS code given in part 2 of Corollary 3.0.1. Results are shown in tables throughout chapter 5.

5.1 FWS Codes in the Hamming Metric

As established in Section 4.1, in the context of Proposition 3.0.1 the Hamming weight function satisfies $D = m = 1$ with $w_H(\mathbb{F}_q^n) = \{1\}$. Thus we have that $|w_H(\mathbb{F}_q^n)| = n$ giving the following definition:

Definition: An $[n, k]_q$-code C is a *Hamming Full Weight Spectrum* code if:

$$|w_H(C)| = n \tag{5.1}$$

Proposition 3.0.1 also gives us the following upper bound on the length of a Hamming-FWS code.

$$|w_H(C)| = n \implies n \leq \frac{q^k - 1}{q - 1}$$

In [2] it was shown that Hamming Full Weight Spectrum codes exist over $\mathbb{F}_q$ for all prime powers q if and only if $n < 2^k$.

5.2 FWS Codes in the Lee Metric

As established in Section 4.2, in the context of Proposition 3.0.1 the Lee weight function satisfies $D = m = \lfloor \frac{q}{2} \rfloor$ with $w_L(\mathbb{F}_q) = \{1, \ldots, \lfloor \frac{q}{2} \rfloor\}$. Thus we have that $|w_L(\mathbb{F}_q^n)| = n \cdot \lfloor \frac{q}{2} \rfloor$ giving the following definition:

Definition: An $[n, k]_q$-code C is a *Lee Full Weight Spectrum Code* if:

$$|w_L(C)| = n \cdot \left\lfloor \frac{q}{2} \right\rfloor \tag{5.2}$$

The following tables show $L_L(n, k, q)$ compared to $n \cdot \lfloor \frac{q}{2} \rfloor$ for the indicated values of k and q. The values in tables 5.1 and 5.2 were found by use of computational software to exhaustively check the Lee weight sets of the span of every set of k linearly independent vectors in $\mathbb{F}_q^n$.

Table 5.1: Results for 1-dimensional 5-ary FWS Codes in The Lee Metric

$q = 5$	$\frac{q^k-1}{q-1} = 1$	$\frac{q^k-1}{2} = 2$		
Length	**Number of Codes**	$\mathbf{L_L(n, 1, 5)}$	$\mathbf{n \cdot \lfloor \frac{q}{2} \rfloor}$	**Notes**
$n = 1$	1	2	2	1 FWS code
$n = 2$	6	2	4	0 FWS codes

Table 5.2: Results for 2-dimensional 5-ary FWS Codes in The Lee Metric

$q = 5$	$\frac{q^k-1}{q-1} = 6$	$\frac{q^k-1}{2} = 12$		
Length	**Number of Codes**	$\mathbf{L_L(n, 2, 5)}$	$\mathbf{n \cdot \lfloor \frac{q}{2} \rfloor}$	**Notes**
$n = 2$	1	4	4	1 FWS code
$n = 3$	31	6	6	6 FWS codes
$n = 4$	806	8	8	16 FWS codes
$n = 5$	20306	8	10	0 FWS codes
$n = 6$	508431	9	12	0 FWS codes
				No $[n, 2]_5$ Lee FWS codes with $n > 4$

The next proposition shows existence of a Lee FWS code with $n = \frac{(\frac{q+1}{2})^k - 1}{(\frac{q+1}{2}) - 1}$ for any positive integer k and any $q > 2$ prime.

Proposition 5.2.1. *Let k be a postive integer and let $\overline{\mathbf{e}}_i$ be the length k column vector with the ith coordinate 1 and 0s elsewhere. Let $q > 2$ be prime and and let G be the following $k \times n$ matrix over $\mathbb{F}_q$. Let $\alpha = (q+1)/2$ and*

$$G = \begin{bmatrix} \overline{\mathbf{e}}_1 & | \overbrace{\overline{\mathbf{e}}_2 \ldots \overline{\mathbf{e}}_2}^{\alpha \text{ times}} | & \ldots & | \overbrace{\overline{\mathbf{e}}_k \ldots \overline{\mathbf{e}}_k}^{\alpha^{k-1} \text{ times}} | \end{bmatrix}$$

The linear code generated by G is a Lee FWS code.

Proof. Let C be the code generated by G and let $\overline{\mathbf{c}}$ be a nonzero element of C. Since q is odd, we wish to show that $|w_L(C)| = n(\frac{q-1}{2})$. By definition, $\overline{\mathbf{c}}$ is equal to the product $(a_1, \ldots, a_k) \cdot G$ for some $a_1, \ldots, a_k \in \mathbb{F}_q$. Use the notation $w_L(a) = |a|$ for scalars $a \in \mathbb{F}_q$. It is clear that

$$w_L(\overline{\mathbf{c}}) = |a_1| + |a_2| \cdot \alpha + \cdots + |a_k| \alpha^{k-1}$$

Since $|a_i| \in \{0, 1, \ldots, (q-1)/2 = \alpha - 1\}$ for $i = 1, \ldots, k$, the right hand side of the above equation expresses $w_L(\overline{\mathbf{c}})$ uniquely as an integer in base α. Note that $n = \frac{\alpha^k - 1}{\alpha - 1}$. By basis representation theorem and by proposition 5.2.1, the scalars $a_1, \ldots, a_k$ can be chosen such that $w_L(\overline{\mathbf{c}})$ is any of the following set $\{1, \ldots, \alpha^k - 1\}$. So $|w_L(C)| = \alpha^k - 1 = n(\alpha - 1) = n(\frac{q-1}{2})$. □

Experimental data provides evidence for the following conjecture. At the time of writing, I do not have an analytical proof.

Conjecture 1. *Let G' be the matrix given in the previous proposition with any number of columns removed without creating an all zero row. Then G' generates a Lee FWS code. Consequently, a length n Lee FWS code exists if $k \leq n \leq \frac{(\frac{q+1}{2})^k - 1}{(\frac{q+1}{2}) - 1}$.*

Corollary 5.2.1. *If Conjecture 1 holds, the maximum length of a Lee FWS code is at least $\frac{(\frac{q+1}{2})^k - 1}{(\frac{q+1}{2}) - 1}$*

5.3 Manhattan FWS Codes

As established in section 4.2, in the context of Propostion 3.0.1, Manhattan Weight satisfies $D = m = q - 1$ with $w_M(\mathbb{F}_q) = \{1, \ldots, q-1\}$. Thus we have that $|w_M(\mathbb{F}_q^n)| = n \cdot (q-1)$ giving the following definition:

Definition: An $[n, k]_q$-code C is a *Manhattan Full Weight Spectrum Code* if:

$$|w_M(C)| = n \cdot (q - 1) \tag{5.3}$$

The following tables show $L_M(n, k, q)$ compared to $n \cdot (q-1)$ for the indicated values of k and q. The values in tables 5.3 and 5.4 were found by use of computational software to exhaustively check the Lee weight sets of the span of every set of k linearly independent vectors in $\mathbb{F}_q^n$.

Table 5.3: Results for 2-dimensional 2-ary Manhattan FWS Codes

$q = 2$	$\frac{q^k-1}{q-1} = 3$	$q^k - 1 = 3$		
Length	**Number of Codes**	$\mathbf{L_M(n, 2, 2)}$	$\mathbf{n \cdot (q-1)}$	**Notes**
$n = 2$	1	2	2	1 FWS code
$n = 3$	7	3	3	3 FWS codes

Table 5.4: Results for 2-dimensional 3-ary Manhattan FWS Codes

$q = 3$	$\frac{q^k-1}{q-1} = 4$	$q^k - 1 = 8$		
Length	**Number of Codes**	$\mathbf{L_M(n, 2, 3)}$	$\mathbf{n \cdot (q-1)}$	**Notes**
$n = 2$	1	4	4	1 FWS code
$n = 3$	13	6	6	3 FWS codes
$n = 4$	130	8	8	4 FWS codes

Table 5.5: Results for 2-dimensional 5-ary Manhattan FWS Codes

$q = 5$	$\frac{q^k-1}{q-1} = 6$	$q^k - 1 = 24$		
Length	**Number of Codes**	$\mathbf{L_M(n, 2, 5)}$	$\mathbf{n \cdot (q-1)}$	**Notes**
$n = 2$	1	8	8	1 FWS code
$n = 3$	31	12	12	3 FWS codes
$n = 4$	806	16	16	4 FWS codes
$n = 5$	20306	20	20	5 FWS codes
$n = 6$	508431	24	24	6 FWS codes

Proposition 5.3.1. *Let k be a postive integer and let $\overline{\mathbf{e}}_i$ be the length k column*

vector with the ith coordinate 1 and 0s elsewhere. Let q be prime and and let G be following $k \times n$ matrix over $\mathbb{F}_q$:

$$G = \left[\overline{\mathbf{e}}_1 \quad | \overbrace{\overline{\mathbf{e}}_2 \dots \overline{\mathbf{e}}_2}^{q \ times} | \quad \dots \quad | \overbrace{\overline{\mathbf{e}}_k \dots \overline{\mathbf{e}}_k}^{q^{k-1} \ times} | \right]$$

Then G generates a Manhattan FWS code.

Proof. See Remark following proof of Proposition 4.3.1. □

Experimental data provides evidence for the following conjecture. At the time of writing, I do not have an analytic proof.

Conjecture 2. *Let G' be the matrix given in the previous proposition with any number of columns removed without creating an all zero row. Then G' generates a Manhattan FWS code. Consequently, a length n Manhattan FWS code exists if $k \leq n \leq \frac{q^k-1}{q-1}$*

Corollary 5.3.1. *Recall that no Manhattan FWS codes exist with $n > \frac{q^k-1}{q-1}$. If Conjecture 2 holds, an $[n, k]_q$ Manhattan FWS code exists if and only if $k \leq n \leq \frac{q^k-1}{q-1}$*

Chapter 6

Conclusions and Open Problem

6.1 Conclusions

With respect to maximum weight spectrum (MWS) codes, we started by identifying a theoretical upper bound on the functions $L_L(k,q)$ and $L_M(k,q)$, and then showed the bound was sharp by constructing a generator matrix for a code which meets the bound with equality. As discussed in Chapter 3, there is a lower bound on the length of MWS codes with respect to a component wise weight function. Perhaps the most natural first question to then ask about MWS codes is "how short can they be?". This question remains open for the Hamming and Lee weight functions, but was answered for the Manhattan weight function.

With respect to full weight spectrum (FWS) codes, the question is "how long can they be?". An obvious lower bound of the length of an FWS code is k, for when $n = k$, C is the ambient space thus will have the same weight set. A theoretical upper bound the length of FWS codes with respect to a competent wise weight function is given in Chapter 3. This bound, $n \leq \frac{q^k-1}{q-1}$, has been shown to be sharp in the case of Manhattan weights. For Hamming weights the maximum length of an FWS code is $n = 2^k - 1$ [2]. In this work, the maximum length of a Lee FWS code is conjectured to be at least $n = \frac{(\frac{q+1}{2}^k)-1}{(\frac{q+1}{2})-1}$.

The main results regarding MWS codes are summarized in the following table. Note that for some values of q and k, the minimum length of a Lee MWS code has been determined to be strictly greater than $\frac{q^k-1}{q-1}$.

Table 6.1: MWS Code Results

Weight Function	$\mathbf{L(k, q)}$	$\mathbf{n_{MWS}(k, q)}$
Hamming Weight	$\frac{q^k-1}{q-1}$ for every prime power q and every $k \in \mathbb{Z}^+$	$\geq \frac{q^k-1}{q-1}$
Lee Weight	$\frac{q^k-1}{2}$ for every prime $q > 2$ and every $k \in \mathbb{Z}^+$	$\geq \frac{q^k-1}{q-1}$
Manhattan Weight	$q^k - 1$ for every prime q and every $k \in \mathbb{Z}^+$	$\frac{q^k-1}{q-1}$

The next table summarizes conditions for the existence of FWS codes.

Table 6.2: Conditions for FWS Codes

Weight Function	Condition for existence of FWS Code
Hamming Weight	if and only if $n < 2^k$, q prime power
Lee Weight	if $k \leq n \leq \frac{(\frac{q+1}{2})^k-1}{(\frac{q+1}{2})-1}$, $q > 2$ prime
Manhattan Weight	if $k \leq n \leq \frac{q^k-1}{q-1}$, $q > 2$ prime

6.2 Open Problem

The following conjecture is an attempt to generalize the results for the length of Hamming, Lee, and Manhattan FWS codes to any componentwise weight function. In the context of Conjecture 3, Hamming-weight corresponds to $D = 1$, wheres the Lee-weight and Manhattan-weight correspond to $D = \lfloor \frac{q}{2} \rfloor$ and $D = q-1$ respectively. Putting the respective D values for each weight function into Conjecture 3 yields results consistent with the results of this thesis.

Determine the truth of the following conjecture.

Conjecture 3. *Let $w(x)$ be a component-wise, integer valued weight function on $\mathbb{F}_q^n$ such that $\max_{x \in \mathbb{F}_q^n} |w(\langle x \rangle)| = D$ and $w(\mathbb{F}_q) = \{1, \ldots, D\}$ for some $D \in \mathbb{Z}$.*

1. *Let $G(\gamma)$ be the following matrix over $\mathbb{F}_q$:*

$$G(\gamma) = \begin{bmatrix} 1 & 0 & \ldots & & & \\ 0 & \underbrace{1 \ldots 1}_{\gamma} & 0 & \ldots & & \\ 0 & \ldots & \underbrace{1 \ldots 1}_{\gamma^2} & 0 & \ldots & \\ \vdots & & & & & \\ & & \ddots & & & \\ 0 & \ldots & & \underbrace{1 \ldots 1}_{\gamma^{k-1}} & & \end{bmatrix}$$

   Then $G(D+1)$ generates a linear code that is FWS with respect to weight function $w(x)$.

2. *Let $n_\gamma = \frac{\gamma^k - 1}{\gamma - 1}$. Then $[n, k]_q$-codes that are FWS with respect to weight function $w(x)$ exist if $k \leq n \leq n_{(D+1)}$.*

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Appendix A

Summary of Results

Table A.1

$(\mathbf{n}, \mathbf{k}, \mathbf{q})$	*Number of Codes*	*Number of H-MWS Codes*	*Number of L-MWS Codes*	*Number of M-MWS Codes*	*Number of L-FWS Codes*	*Number of M-FWS Codes*	$\boldsymbol{L_H(n, k, q)}$	$\boldsymbol{L_L(n, k, q)}$	$\boldsymbol{L_M(n, k, q)}$
$(1, 1, 2)$	1			1					1
$(1, 1, 3)$	1			1					2
$(1, 1, 5)$	1		1	1	1			2	4
$(2, 1, 2)$	3			3					1
$(2, 1, 3)$	4			3					2
$(2, 1, 5)$	6		4	5	0			2	4
$(2, 2, 2)$	1	0		0		1	2		2
$(2, 2, 3)$	1	0		0		1	2		4

$(\mathbf{n}, \mathbf{k}, \mathbf{q})$	*Number of Codes*	*Number of H-MWS Codes*	*Number of L-MWS Codes*	*Number of M-MWS Codes*	*Number of L-FWS Codes*	*Number of M-FWS Codes*	$\boldsymbol{L_H(n,k,q)}$	$\boldsymbol{L_L(n,k,q)}$	$\boldsymbol{L_M(n,k,q)}$
$(2,2,5)$	1	0	0	0	1	1	2	4	8
$(3,2,2)$	7	3		0		3	3		2
$(3,2,3)$	13	0		0		3	3		6
$(3,2,5)$	31	0	0	0	6	3	3	6	12
$(4,2,2)$	35	16					3		
$(4,2,3)$	130	0		4		4	3		8
$(4,2,5)$	806	0	0	0	16	4	3	8	16
$(5,2,2)$	155	65					3		
$(5,2,3)$	1210	0		35			3		8
$(5,2,5)$	20306	0	0	0	0	5	4	8	20
$(6,2,2)$	651	291					3		
$(6,2,3)$	11011	1920		276			4		8
$(6,2,5)$	508431	0	0	6	0		4	9	24
$(7,2,3)$	99463	20160					4		
$(7,2,5)$	12714681	n/a	0				n/a	9	
$(8,2,3)$	896260	n/a					n/a		

Appendix B

Sample Code

Code used in this thesis is written in SageMath V.9.2 (https://www.sagemath.org) using Python 3.7.7. and is available on Github at the following repository address: https://github.com/BenMorine/Thesis-Code.git

Following is sample code for computing the maximum number of Lee weights in a length 4, 2-dimensional, 5-ary code $L_L(4, 5, 2)$, as included in Table 4.5 and Table 5.2.

Sample Code for determining $L_L(4, 2, 5)$:

First, the parameters n, k, and q are declared. Then functions which compute the Lee weight of a vector and the Lee weight set of a collection of vectors are defined.


```
{
"cells": [
{
"cell_type": "code",
"execution_count": 1,
"metadata": {},
"outputs": [],
"source": [
```

```
"k=2\n",
"q=5\n",
"n=4\n",
"F=FiniteField(q)\n",
"def Lee(x):\n",
"    return sum([ZZ(min(xi,q-xi)) for xi in x])\n",
"def W_Lee(X):\n",
"    return {Lee(x) for x in X}\n",
"from sage.combinat.cartesian_product import CartesianProduct_iters"
]
},
{
```

Now we check the span of every pair of vectors from $\mathbb{F}_5^4$ and store each 2-dimensional span in a new set called "W".

```
"cell_type": "code",
"execution_count": 2,
"metadata": {},
"outputs": [],
"source": [
"V={span([v,u]) for (v,u) in CartesianProduct_iters(F^n,F^n)}"
]
},
{
"cell_type": "code",
"execution_count": 4,
"metadata": {},
"outputs": [],
```

"source": [
"W=set()\n",
"for X in V:\n",
"   if X.dimension()==k:\n",
"   W.add(X)"
]
},
{
"cell_type": "code",
"execution_count": 5,
"metadata": {},
"outputs": [],
"source": [

Now we define the set $A = \{|w_L(C)| \mid C \text{ is a } [4,2]_5 \text{ code}\}$ whereby we obtain $L_L(4,2,5)$ as $\max A$.

"A={Set(W_Lee(Y)-{0}).cardinality() for Y in W}"
]
},
{
"cell_type": "code",
"execution_count": 6,
"metadata": {},
"outputs": [
{
"data": {
"text/plain": [

"{2, 3, 4, 5, 6, 8}"
]
},
"execution_count": 6,
"metadata": {},
"output_type": "execute_result"
}
],
"source": [
"A"
]
},
{
"cell_type": "code",
"execution_count": 6,
"metadata": {},
"outputs": [
{
"data": {

Here we show explicitly all the Lee weight sets of $[4, 2]_5$-codes.

"text/plain": [
"{{3, 4, 5, 6, 7, 8},\n",
" {2, 4, 5, 6, 7},\n",
" {8, 4, 5, 6},\n",
" {8, 2, 4, 6},\n",
" {1, 2, 3, 4, 5, 6},\n",
" {2, 3, 4, 5, 6},\n",

" {1, 2, 3, 4},\n",
" {3, 4, 5, 6, 7},\n",
" {1, 2, 3, 4, 5},\n",
" {2, 3, 4, 5, 6, 7},\n",
" {2, 4, 5},\n",
" {3, 6},\n",
" {1, 2, 3, 4, 5, 6, 7, 8},\n",
" {2, 4, 5, 7, 8},\n",
" {1, 2, 4, 5, 6, 7},\n",
" {2, 3, 4, 5, 7},\n",
" {3, 4, 5, 6}}"
]
},
"execution_count": 6,
"metadata": {},
"output_type": "execute_result"
}
],
"source": [
"H={Set(W_Lee(Q)-{0}) for Q in W}\n",
"H"
]
},
{
"cell_type": "code",
"execution_count": 7,
"metadata": {},

"outputs": [
{
"data": {
"text/plain": [
"806"
]
},
"execution_count": 7,
"metadata": {},
"output_type": "execute_result"
}
],
"source": [
"Set(W).cardinality()"
]
},
{
"cell_type": "code",
"execution_count": 18,
"metadata": {},
"outputs": [
{
"data": {
"text/plain": [
"16"
]
},

"execution_count": 18,
"metadata": {},
"output_type": "execute_result"
}
],
"source": [

The next section of code shows us how many $[4, 2]_5$-codes there are with a specified Lee weight set cardinality.

```
"E=set()\n",
"for P in W:\n",
"    if Set(W_Lee(P)-{0}).cardinality()==8:\n",
"        E.add(P)\n",
"Set(E).cardinality()"
```

]
},
{
"cell_type": "code",
"execution_count": null,
"metadata": {},
"outputs": [],
"source": []
}
],
"metadata": {
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"language": "sage",

"name": "sagemath"
},
"language_info": {
"codemirror_mode": {
"name": "ipython",
"version": 3
},
"file_extension": ".py",
"mimetype": "text/x-python",
"name": "python",
"nbconvert_exporter": "python",
"pygments_lexer": "ipython3",
"version": "3.7.7"
}
},
"nbformat": 4,
"nbformat_minor": 4
}

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