Quasi-Free Actions on Graph Algebras: KMS States and the Structure of Crossed Products

by

Christopher Chlebovec

Master of Science, Lakehead University, 2010
Honours Bachelor of Science, Lakehead University, 2008

A Dissertation Submitted in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

in the Graduate Academic Unit of Mathematics & Statistics

Supervisors:	Dan Kucerovsky, PhD, Dept. of Math & Stats
	Andrew Dean, PhD, Lakehead University, Adjunct Prof., Dept. of Math & Stats
Examining Board:	Tim Alderson, PhD, Dept. of Math & Stats
	David Bremner PhD, Faculty of Computer Science
	Rodney Cooper, PhD, Faculty of Computer Science
External Examiner:	Alex Kumjian, PhD, Dept. of Math & Stats, University of Nevada

**This dissertation is accepted by the
Dean of Graduate Studies**

THE UNIVERSITY OF NEW BRUNSWICK

June, 2016

Abstract

This dissertation deals with the study of C^*-dynamical systems. A C^*-dynamical system consists of a space called a C^*-algebra along with an action of a group on it. The systems that are the object of study here are ones that consist of C^*-algebras obtained from directed graphs, along with actions, known as quasi-free actions, which depend on a labeling map.

In physics, C^*-dynamical systems are used to model physical systems. In these models, the states of the system are described by certain linear functionals on the algebra. The main results of this dissertation deal with the study of Kubo-Martin-Schwinger (KMS) states. These are the functionals that describe the equilibrium states of the physical system.

Within this dissertation, a complete characterization of KMS states is given for C^*-dynamical systems where the C^*-algebra comes from a finite graph and the real action is quasi-free. This characterization provides a framework to generalize results for a specific example of a quasi-free action known as the gauge action. More specifically, an explicit construction of all KMS states above a certain critical inverse temperature is provided, as well as a complete description of the KMS states when the corresponding graph has a certain strongly connected subgraph.

In the study of C^*-dynamical systems a certain algebra, called a crossed product, is used to describe the representations of the system. The structure of crossed products of graph algebras by quasi-free actions is also investigated in this thesis.

For certain graph algebras, it is shown that some of the crossed products have nice inductive limit structures, extending known results for crossed products of Cuntz algebras by quasi-free actions.

Acknowledgments

First and foremost, I wish to thank my advisors Dr. Andrew J. Dean and Dr. Dan Kucerovsky for their guidance and support throughout my PhD studies. I wish to also extend my gratitude to the faculty, staff, and students with whom I've had the pleasure of coming into contact at UNB.

On a personal level, I am grateful to my parents and family for their unwavering love and support and always encouraging me to pursue my interests. Last, but certainly not least, I would like to thank my wife, Kristen, for her endless love and encouragement, and for being by my side throughout.

Table of Contents

List of Notation

Standard notation

A	Typical abstract C^*-algebra
Aut A	Automorphism group of A.
$\mathcal{H}$	A Hilbert space
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+^*$	Positive real numbers equipped with multiplication
$\mathbb{C}$	The set of complex numbers
$\mathbb{T}$	The unit circle in $\mathbb{C}$
$\rho(A)$	Spectral radius of A
$B(\mathcal{H})$	The set of all bounded linear operators from $\mathcal{H}$ to itself
$U(\mathcal{H})$	The set of all unitary operators in $B(\mathcal{H})$
$C_0(X)$	The set of complex valued continuous functions on X vanishing at infinity
$C_0(X, A)$	The set of A-valued continuous functions on X vanishing at infinity
$C_c(X, A)$	The set of A-valued continuous functions on X with compact support

---

Some Preliminaries

(A, G, α)	C^*-dynamical system	p. 18
$A \rtimes_\alpha G$	Crossed product of (A, G, α)	p. 19
$K_0(A), K_1(A)$	K_0 and K_1 groups of a C^*-algebra A	p. 16
O_A	Cuntz-Krieger algebra	p. 3
O_n	Cuntz algebra	p. 3

Graph theory and Graph C*-algebras

$E = (E^0, E^1, r, s)$	A directed graph	p. 7

E^0	The set of all vertices in a graph E	p. 7
E^1	The set of all vertices in a graph E	p. 7
r	A map $r : E^1 \to E^0$ called a range map	p. 7
s	A map $s : E^1 \to E^0$ called a source map	p. 7
E^0_{reg}	The set of all regular vertices of a graph E	p. 8
E^0_{sinks}	The set of all sinks of a graph E	p. 7
E^n	The set of all paths of length n in E	p. 7
E^*	The set of all paths in E	p. 7
vE^nw	$\{\mu \in E^n : s(\mu) = v \text{ and } r(\mu) = w\}$	p. 7
vE^n	$\{\mu \in E^n : s(\mu) = v\}$	p. 7
E^nw	$\{\mu \in E^n : r(\mu) = w\}$	p. 7
$E\backslash H$	$(E^0\backslash H, E^1\backslash r^{-1}(H), r, s)$	p. 38
$\widehat{E}$	Dual graph	p. 12
A_E	Vertex matrix of E	p. 8
B_E	Edge matrix of E	p. 8
$C^*(E)$	Graph C^*-algebra associated to E	p. 11
$\mathcal{T}C^*(E)$	Toeplitz algebra associated to E	p. 22
$C^*(E_{\mathcal{T}})$	Graph algebra that is isomorphic to the Toeplitz algebra $\mathcal{T}(E)$	p. 26
s_e	A typical partial isometry associated to an edge $e \in E^1$	p. 8
p_v	A typical projection associated to a vertex $v \in E^0$	p. 8

Quasi-free actions and related notation

ω, λ, c	Typical labeling maps ($E^1 \to \mathbb{R}$)	p. 20
$\widehat{\omega}$	Labeling map on the dual graph defined by $\widehat{\omega}(ef) = \omega(e)$ for all $ef \in \widehat{E}^1$	p. 40
γ	Typical notation for the gauge action of $\mathbb{T}$ and $\mathbb{R}$	p. 20
α^ω	A quasi-free action corresponding to ω	p. 20
$\eta^{\widehat{\omega}}$	A quasi-free action corresponding to $\widehat{\omega}$	p. 40
$C_{\beta,E,\omega}$	A specific $\lvert E^0\rvert \times \lvert E^0\rvert$ matrix	p. 28
$R_\beta, \tilde{R}_\beta$	Specific $(\lvert E^0_{\text{reg}}\rvert + 1) \times \lvert E^0\rvert$ matrices	p. 30, 31
$C^*(E)^\gamma$	Fixed point subalgebra of $C^*(E)$	p. 28
$(C^*(E), \alpha^\omega)$	The C^*-dynamical system consisting of a quasi-free action acting on a graph algebra	p. 20
K_{β,α^ω}	The set of all KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$	p. 29
$\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$	The set of all KMS$_\beta$ states of $(C^*(\widehat{E}), \eta^{\widehat{\omega}})$	p. 40
Property P_β	Vectors that satisfy a certain matrix equation	p. 28, 40
L_{β,α^ω}	$\{m = (m_v)_{v\in E^0} : m \text{ satisfies Property } P_\beta \text{ on } E^0\}$	p. 29
$L_{\beta,\eta^{\widehat{\omega}}}$	$\{y = (y_e)_{v\in E^1} : m \text{ satisfies Property } P_\beta \text{ on } E^1\}$	p. 40
β_c	Critical inverse temperature	p. 31

Skew-product graphs and related notation

$E \times_c G$	Skew-product graph (G is a countable group)	p. 52
$O_n(k_1, k_2, \ldots, k_n)$	Skew-product graph $C^*(O_n \times_c \mathbb{Z})$ with distinct labels $k_1, \ldots, k_n$, where $k_n > k_{n-1} > \cdots > k_1 > 0$ and $\gcd(k_1, \ldots, k_n) = 1$	p. 66
$A_{(k_1,k_2,\ldots,k_n)}$	Leslie matrix	p. 66
L^m	A certain set equipped with the product topology on $\mathbb{R}^m$	p. 57
L^∞	A certain set equipped with the product topology on $\mathbb{R}^\infty$	p. 59

Glossary

AF-algebra - A C^*-algebra is an *AF-algebra* if it can be written as the closure of the union of an increasing sequence of finite-dimensional C^*-algebras, or equivalently, if it is an inductive limit of a sequence of finite-dimensional C^*-algebras. A C^*-algebra A is *AF-embeddable* if there is an injective $*$-homomorphism that maps A into an AF-algebra.

C*-algebra - A Banach algebra is a complex algebra which is a Banach space under a norm which is submultiplicative ($||ab|| \leq ||a||||b||$ for all $a, b \in A$). An involution on a Banach algebra A is a conjugate-linear map $a \mapsto a^*$ on A, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. A Banach $*$-algebra is a Banach algebra with involution. A *C^*-algebra* is a Banach $*$-algebra A that satisfies the C^*-axiom: $||a^*a||^2 = ||a||^2$. A C^*-algebra is said to be *simple* if 0 and A are its only closed two-sided ideals.

Hereditary subalgebra - Let A be C^*-algebra. A C^*-subalgebra B of A is called a *hereditary subalgebra* if $bab' \in B$ for all $a \in A$ and $b, b' \in B$.

Partial isometry - An element s is a *partial isometry* if ss^* is a projection (s^*s will also be a projection). The projection ss^* is called the range projection and s^*s is called the source projection.

Projections - A element p in a C^*-algebra A is called a *projection* if $p = p^* = p^2$. Two projections in a C^*-algebra A are called *orthogonal* if their product is zero. Two projections p and q in A are *equivalent*, written $p \sim q$, if there is a partial isometry $s \in A$ with $s^*s = p$ and $ss^* = q$. A projection p is equivalent to a subprojection of q if there is a partial isometry $s \in A$ with $s^*s = p$ and $ss^* \leq q$. A projection p is called infinite if it is equivalent to a proper subprojection.

Purely infinite - A C^*-algebra is called *purely infinite* if every hereditary C^*-subalgebra contains an infinite projection.

Stably projectionless - A C^*-algebra A is *stably projectionless* if $A \otimes \mathcal{K}$ contains no projections, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space.

State - A positive linear functional of norm one.

Trace of an operator - Let $\mathcal{H}$ is a Hilbert space and $0 \leq T \in B(\mathcal{H})$. Define the *trace* of T, denoted $\mathrm{Tr}(T)$, to be $\sum_i \langle T\eta_i, \eta_i \rangle \in [0, \infty]$, where $\{\eta_i\}_i$ is an orthonormal basis for $\mathcal{H}$.

Unitary - A element u in a C^*-algebra A is called a *unitary* if $u^*u = uu^* = 1$.

Chapter 1

Motivation and Introduction

C^*-algebras are prominent objects of study in the area of functional analysis. They connect to multiple disciplines, including quantum mechanics, dynamical systems, classification theory, spectral theory, noncommutative topology, harmonic analysis, and many others.

The notion of a C^*-dynamical system has evolved from its classical counterpart. A classical dynamical system consists of points in a space X, which are called states, and a time evolution that describes the states of the system at a given time t. Mathematically, the time evolution in a classical dynamical system is given by a continuous action of the real numbers on a state space X. By the Gelfand-Naimark Theorem, every commutative C^*-algebra is of the form $C_0(X)$, and an action of a locally compact group G on X gives rise to an action of G by automorphisms on $C_0(X)$. In general, a C^*-dynamical system is defined as the noncommuative analogue of the above phenomena. Namely, a C^*-dynamical system, denoted (A, G, τ), consists of a C^*-algebra A, a locally compact group G and a strongly continuous action τ from G into the automorphism group $\mathrm{Aut}A$ of A.

C^*-dynamical systems have provided examples in many areas of mathematics, in particular, quantum statistical mechanics. The modelling of a quantum system can be written as the C^*-dynamical system $(A, \mathbb{R}, \tau)$, where the elements of a C^*-algebra A are used to describe the observables of a quantum system, and an action of the reals describes the time evolution of the system. An important goal of the theory of C^*-dynamical systems is the description of the KMS states. The notion of KMS states evolved from the Gibbs states of a finite dimensional quantum system $(A, \mathbb{R}, \tau)$. That is, where A is a finite-dimensional C^*-algebra, $\tau : \mathbb{R} \to \mathrm{Aut}A$ is the action defined by $\tau_t(a) = e^{itH}ae^{-itH}$ for each $a \in A$, and H is a self-adjoint linear operator from a finite dimensional Hilbert space to itself called the Hamiltonian of the system. For each $\beta \geq 0$, a β-Gibbs state is defined as

$$\phi(a) = \frac{\mathrm{Tr}(e^{-\beta H}a)}{\mathrm{Tr}(e^{-\beta H})},$$

where Tr denotes the trace of an operator and β is called the inverse temperature. These states are known as equilibrium states, as they satisfy the important property of being invariant under the time evolution of the system. That is,

$$\phi(\tau_t(a)) = \phi(a),$$

for each $a \in A$. Kubo, Martin, and Schwinger extended the notion of equilibrium states to make sense in the context of infinite dimensional quantum systems. In fact, the KMS_β states are exactly the β-Gibbs states when restricted to finite dimensional systems. These historical facts can be found in [5] and the references therein.

In this thesis, the C^*-algebras that are considered are ones that are defined from directed graphs. These C^*-algebras are called graph algebras, where the vertices and edges in the graph correspond to elements of the C^*-algebra and the directed edges determine the relationship of these elements, via algebraic operations. In particular, the edges correspond to partial isometries with mutually orthogonal ranges and the vertices correspond to mutually orthogonal projections.

The notion of a graph algebra emerged from the study of the Cuntz-Krieger algebra O_A. This is defined as the universal C^*-algebra generated by a collection of partial isometries $s_1, s_2, \dots, s_n$ with mutually orthogonal ranges ($s_i s_i^* s_j s_j^* \neq 0$ for $i \neq j$) that satisfy the following relations:

$$s_i^* s_i = \sum_{i=1}^{n} A(i,j) s_j s_j^*,$$

where A is an $n \times n$ $\{0,1\}$-matrix in which every row and column is nonzero [12]. Shortly after [12] appeared, Enomoto and Watatani [20] found a connection between Cuntz-Krieger algebras and directed graphs, where A arises as an adjacency matrix of a directed graph. The associated directed graph contains n vertices with an edge from the i^{th} vertex to the j^{th} if $A(i,j) = 1$. The Cuntz-Krieger algebras are the C^*-algebras of finite directed graphs with no sinks (every vertex emits at least one edge) and no sources (every vertex receives at least one edge). The Cuntz algebra, denoted by O_n, is a special case of a Cuntz-Krieger algebra obtained by having $A(i,j) = 1$ for all $i, j \in \{1, 2, \dots, n\}$.

In 1997, Kumjian, Pask, Raeburn and Renault realized the Cuntz-Krieger algebra O_A as a groupoid C^*-algebra which was determined from a directed graph whose edge matrix was A [39]. The directed graphs that were considered were locally finite graphs (each vertex emits and receives finitely many edges). In 1998, the definition was extended to include an even larger family of graphs called row-finite graphs; that is, graphs in which each vertex emits at most finitely many edges [38]. These graphs included sinks and sources and were defined without the presence of a groupoid structure. Kumjian, Pask and Raeburn showed that if E is a graph with no loops, then the corresponding graph algebra is approximately finite-dimensional (AF). Furthermore, they also showed that simple graph algebras are either AF or purely infinite. Graph algebras are particularly nice objects of study since properties of the C^*-algebra can be deciphered from the structure of its corresponding graph.

Given a graph E, the graph algebra corresponding to E is denoted by $C^*(E)$. It is generated by a collection $\{s_e, p_v\}$, called a Cuntz-Krieger E-family, where s_e are the partial isometries for each edge e and p_v are the projections for each vertex v in the graph E (see Definition 2.1.1). If E^1 is the set of all edges of the graph E and $\omega : E^1 \to \mathbb{R}$ gives a labeling of each edge by a real number, then an action α^ω of the reals on the graph algebra $C^*(E)$ can be defined by $\alpha_t^\omega(s_e) = e^{i\omega(e)t}s_e$ for all $e \in E^1$ and $\alpha_t^\omega(p_v) = p_v$ for all $v \in E^0$. Such an action is called a quasi-free action of the reals. When $\omega(e) = 1$ for all edges $e \in E^1$, the quasi-free action α^ω reduces to the gauge action of $\mathbb{R}$. The study of the C^*-dynamical system $(C^*(E), \alpha^\omega, \mathbb{R})$ and, in particular, its KMS states and the structure of the crossed product $C^*(E) \rtimes_{\alpha^\omega} \mathbb{R}$, are the objects of this thesis.

The notion of KMS states stems from quantum statistical mechanics, with the study of equilibrium states and its extension to infinite quantum systems. For the gauge action γ of $\mathbb{R}$, Enomoto, Fujii and Watatani [19] gave a description of the KMS states of the Cuntz-Krieger algebra O_A, in terms of the eigenvalues of A. In particular, they showed that when A is an irreducible matrix, there exists a unique KMS state. The KMS state has inverse temperature $\ln \rho(A)$, where $\rho(A)$ is the spectral radius of A (or, equivalently, the Perron-Frobenius eigenvalue of A).

Exel and Laca [22] extended the results in [19] to certain quasi-free actions, where the labels are all positive and A is a finite matrix with no zero rows or columns. When A is an irreducible matrix, they gave a complete description of the KMS states for the Toeplitz-Cuntz-Krieger algebra $\mathcal{T}_A$. Among their results, they showed that, at a critical inverse temperature $\beta_c > 0$, there exists a unique KMS_{β_c} state of $\mathcal{T}_A$. This state factors through O_A and is the only KMS state of O_A. Zacharias [58] also showed that there exists a unique KMS state of O_A, but without the use of the Toeplitz-Cuntz-Krieger algebra. The unique β_c satisfies $\rho(D_{\beta_c}A) = 1$, where D_{β_c} is a diagonal matrix and each diagonal entry is of the form $e^{-\beta_c \lambda}$ for some label $\lambda > 0$.

More recently, there has been interest in the investigation of KMS states of C^*-algebras that are constructed from directed graphs. In [30, 29, 28], finite graphs were analyzed and, in [8, 14], emphasis was towards infinite graphs. In [29] and [28], an Huef, Laca, Raeburn and Sims gave considerable insight into KMS states on the C^*-algebras of finite graphs for the gauge action of $\mathbb{R}$; their work consisted of studying KMS states on the Toeplitz algebra $\mathcal{T}C^*(E)$ for a finite graph E. In [29], they gave an explicit description of all KMS_β states, when β is above a certain critical inverse temperature β_c. It was also shown that if E is a strongly connected graph, then there is a unique KMS_{β_c} state of $\mathcal{T}C^*(E)$ that factors through $C^*(E)$. In [28], they continued their analysis, with emphasis on graph algebras having reducible vertex matrices, by looking at the strongly connected components of a finite graph E and their interactions.

In addition to studying KMS states, there has been significant interest in crossed products of graph algebras by quasi-free automorphisms. In particular, when the graph algebra is a Cuntz algebra, Kishimoto [34] showed that that the crossed product $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is simple if and only if one of the following two cases occur:

1. All the labels λ_k are of the same sign and $\{\lambda_1, \ldots, \lambda_n\}$ generate $\mathbb{R}$ as a closed group.

2. The closed subsemigroup generated by all λ_k is $\mathbb{R}$.

In the first case, the crossed product is stably projectionless [35] and in the second case, it is purely infinite [36]. In [33], Katsura completely described the ideal structure of crossed products of O_n or O_∞, giving another proof of Kishimoto's simplicity results. As an extension, Elliott and Fang [18] investigated the ideal structure and simplicity of the crossed product $C^*(E) \rtimes \mathbb{R}$, when E is a row-finite graph without sinks. In [32], a sufficient condition was obtained for the AF-embeddability of a crossed product of O_n and O_∞ by quasi-free actions. Along with being stably projectionless, $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable in case 1. The AF-embeddability of crossed products of certain graph C^*-algebras by quasi-free actions in [23] shows that the methods used by Katsura are restrictive. Dean [15] showed that for λ in a dense G_δ subset of the positive multi-indices, the crossed product of $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ can be written as an inductive limit of non-unital 1-dimensional NCCW-complexes. Based on the construction, the crossed products will be AF-embeddable in this case.

This thesis is organized in the following manner. Chapter 2 contains definitions and preliminaries. Chapter 3 deals with extending the results from [29] to quasi-free actions. The definition of the Toeplitz algebra, as well as the main results from [29], are stated at the beginning of this chapter. Theorem 3.3.1 gives a characterization of KMS states for quasi-free actions on graph algebras, which allows subsequent results to involve the graph algebra $C^*(E)$ directly. Then, as a consequence, we use Proposition 3.2.2 to recover the results for the Toeplitz algebra $\mathcal{T}C^*(E)$. The characterization in Theorem 3.3.1 is also used to compute the KMS states for specific examples. In Chapter 4, we extend the results in [15] to a large class of graph algebras, expressing the structure of the crossed product as an inductive limit of non-unital 1-dimensional NCCW-complexes. Also, the K-theory is calculated for the circle crossed products of certain periodic quasi-free actions. As a consequence of the results on KMS states, the crossed product $C^*(E) \rtimes_{\alpha^\omega} \mathbb{R}$ is stably projectionless, when E is a strongly connected finite graph that is not a single cycle. Lastly, Chapter 5 contains a conclusion summarizing the main results of this thesis and a discussion that explores some interesting questions for future research.

Chapter 2

Preliminaries

This chapter will provide relevant background information that is needed for this thesis.

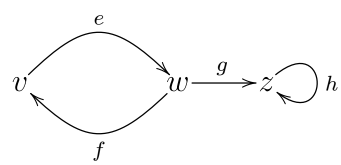
2.1 Graph Algebras

The definitions and terminology for directed graphs are given below and can be found in [56, p. 3] and [46, pp. 5–6].

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0 and E^1 of vertices and edges, respectively, with range and source maps $r, s : E^1 \longrightarrow E^0$. A directed graph $E = (E^0, E^1, r, s)$ is called *finite* if both E^0 and E^1 are finite and is called *row-finite* if $|s^{-1}(v)| < \infty$ for all $v \in E^0$. A *path* of length $n \geq 1$ is a finite sequence of edges $\mu := \mu_1\mu_2 \cdots \mu_n$ with $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i \leq n-1$. We regard vertices as paths of length 0. For $n \geq 0$, we let E^n denote the set of all paths of length n and define $E^* := \bigcup_{n\geq 0} E^n$. The range and source maps extend to E^* in a natural way. For vertices v and w, we define vE^nw to be the set $\{\mu \in E^n : s(\mu) = v \text{ and } r(\mu) = w\}$. A *cycle* is a path with its range and source equal; namely, a path $\mu := \mu_1\mu_2 \cdots \mu_n$ is a cycle provided that $r(\mu_n) = s(\mu_1)$. A vertex that does not emit an edge is called a *sink* and we write E^0_{sinks} for the set of all sinks in E^0. A vertex that emits at least one edge, but not infinitely many edges, is called a *regular vertex* and we denote by E^0_{reg} the set of all regular vertices in E^0. Two important matrices that are associated to a graph $E = (E^0, E^1, r, s)$ are the *vertex matrix*, also known as the *adjacency matrix*, and the *edge matrix*. The vertex matrix is the matrix $A_E \in M_{E^0}(\mathbb{N})$ such that $A_E(v, w) = |vE^1w|$ and the edge matrix is the matrix $B_E \in M_{E^1}(\mathbb{N})$ such that

$$B_E(e, f) = \begin{cases} 1 & \text{if } ef \text{ is a path} \\ 0 & \text{if } ef \text{ is not a path.} \end{cases}$$

Example 2.1.1.



The above directed graph gives: $E^0 = \{v, w, z\}$, $E^1 = \{e, f, g, h\}$ and range and source maps that satisfy $r(f) = s(e) = v$, $r(e) = s(g) = s(f) = w$, and $r(g) = r(h) = s(h) = z$. The vertex matrix A and edge matrix B for the graph E above is defined as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following definition was given in [38, p. 162] for row-finite graphs and in [25, Definition 1] for arbitrary graphs.

Definition 2.1.1. If $E = (E^0, E^1, r, s)$ is a graph, then a Cuntz-Krieger E-family in a C^* -algebra is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges that satisfy the following Cuntz-Krieger relations:

- (CK1) $s_e^* s_e = p_{r(e)}$
- (CK2) $p_v = \sum_{\{e \in E^1 : s(e)=v\}} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$, and
- (CK3) $s_e s_e^* \leq p_{s(e)}$.

We also define, for a path $\mu := \mu_1 \mu_2 \cdots \mu_n$, $s_\mu := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}$ and $s_v = p_v$ for all $v \in E^0$. One can easily show that if μ is a path, then s_μ is a partial isometry with initial projection $p_{r(\mu)}$; otherwise, s_μ is zero.

Given words from the symbols $\{s_e, s_f^*\}_{e,f \in E^1}$, we may use the Cuntz-Krieger relations to collapse them to products of the form $s_\mu s_\nu^*$ for $\mu, \nu \in E^*$, with $r(\mu) = r(\nu)$, via the formula

$$s_\alpha^* s_\beta = \begin{cases} s_{\beta'} & \text{if } \beta = \alpha\beta' \\ p_{r(\alpha)} & \text{if } \alpha = \beta \\ s_{\alpha'}^* & \text{if } \alpha = \beta\alpha' \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the family $\{s_\mu s_\nu^*\}$ is closed under multiplication and involution; thus,

$$C^*(s,p) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}.$$

A directed graph may give rise to Cuntz-Krieger families that generate non-isomorphic C^*-algebras. Thus, we need the notion of the "universal graph C^*-algebra" generated by a Cuntz-Krieger E-Family.

Proposition 2.1.1. [46, Proposition 1.21 on p. 13] *For a row-finite graph E, there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s,p\}$ such that for every Cuntz-Krieger E-family $\{t,q\}$ in a C^*-algebra B, there is a homomorphism $\pi_{t,q} : C^*(E) \longrightarrow B$, satisfying $\pi_{t,q}(s_e) = t_e$ for every $e \in E^1$ and $\pi_{t,q}(p_v) = q_v$ for every $v \in E^0$.*

Proof. Let $V = \{\sum z_{\mu,\nu} d_{\mu,\nu} : \mu, \nu \in E^*, r(\mu) = r(\nu)\}$ be the complex vector space of all formal sums, where all but finitely many of the complex coefficients $z_{\mu,\nu}$ in each sum are zero and $d_{\mu,\nu}$ are just formal symbols. We have that V becomes a $*$-algebra with $d^*_{\mu,\nu} = d_{\nu,\mu}$ and

$$d_{\mu,\nu} d_{\alpha,\beta} = \begin{cases} d_{\mu\alpha',\beta} & \text{if } \alpha = \nu\alpha' \\ d_{\mu,\beta\nu'} & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

For every Cuntz-Krieger E-family $\{S, P\}$ on a Hilbert space $\mathcal{H}$, there is a $*$-representation $\pi_{S,P}$ of V on $\mathcal{H}$ with $\pi_{S,P}(d_{\mu,\nu}) = S_\mu S^*_\nu$. We have that

$$||a||_1 := \sup\{||\pi_{S,P}(a)|| : \{S, P\} \text{ is a Cuntz-Krieger } E\text{-family}\}$$

is an algebra seminorm satisfying $||a^*a||_1 = ||a||_1^2$. Let I be the $*$-ideal $\{u \in V : ||u||_1 = 0\}$. Then, $V_0 = V/I$ is a $*$-algebra, and the quotient norm $||\cdot||_0$ is a C^*-norm, so the completion $\overline{V_0}$ is a C^*-algebra. Each $\pi_{S,P}$ is $||\cdot||_0$-continuous, and hence extends uniquely to a representation of $\overline{V_0}$.

Finally, take $C^*(E) = \overline{V_0}$ and note that $s_e := d_{e,r(e)}$ and $p_v := d_{v,v}$ form a Cuntz-Krieger E-family, which generates V_0. Then, let $\rho : B \to B(\mathcal{H})$, and take $\pi_{t,q} := \rho^{-1} \circ \pi_{\rho(t),\rho(q)}$. □

As a consequence of the above proposition, we can associate to each graph E, a unique C^*-algebra in the following sense:

Corollary 2.1.1. [46, Corollary 1.22 on p.13] *Suppose E is a row-finite directed graph, and C is a C^*-algebra generated by a Cuntz-Krieger E-family $\{w, r\}$ such that for every Cuntz-Krieger E-family $\{t, q\}$ in a C^*-algebra B, there is a homomorphism $\rho_{t,q} : C \to B$ satisfying $\rho_{t,q}(w_e) = t_e$ for every $e \in E^1$ and $\rho_{t,q}(r_v) = q_v$ for every $v \in E^0$. Then, there is an isomorphism ϕ of $C^*(E)$ onto C such that $\phi(s_e) = w_e$ for* *every $e \in E^1$ and $\phi(p_v) = r_v$ for every $v \in E^0$.*

The existence of $C^*(E)$ was shown in [38, Theorem 1.2] for row-finite graphs and in [25, Definition 1] for arbitrary graphs.

Definition 2.1.2. Let $E = (E^0, E^1, r, s)$ be a graph. The **graph $\mathbf{C^*}$-algebra** (or simply, the graph algebra) of E is the C^*-algebra generated by the Cuntz-Krieger E-family that satisfies the universal property and is denoted by $C^*(E)$.

Remark 2.1.1. The results from some papers (for example, [14] and [29]) use a convention that is different from the one used in this thesis. In the alternative convention, a path $e_1e_2\cdots e_n$ has source $s(e_n)$ and range $r(e_1)$. In this case, the partial isometries in the corresponding graph algebra go in the same direction as the arrows in the graph. The results that are cited from these papers are adapted to be consistent with the convention used in this thesis.

2.1.1 Ideal Structure

A subset H of E^0 is *hereditary* if, for any $e \in E^1$, we have $s(e) \in H$ implies $r(e) \in H$. A hereditary set H is *saturated* if, whenever $v \in E^0$ is a regular vertex with $r(vE^1) \subseteq H$, then $v \in H$. If $H \subseteq E^0$ is a hereditary set, the *saturation* of H is the smallest saturated subset $\overline{H}$ of E^0 containing H. If $\gamma : \mathbb{T} \to \operatorname{Aut} A$ is the gauge action, then an ideal I in a C^*-algebra A is said to be *gauge-invariant* if $\gamma_z(a) \in I$ for all $a \in A$ and $z \in \mathbb{T}$. In [2, Theorem 4.1], it was shown that there is a bijective correspondence between the gauge-invariant ideals in $C^*(E)$ and the saturated hereditary subsets of E^0 when E is a row-finite graph.

2.1.2 Uniqueness Theorems

The following uniqueness theorems given below have proved to be very useful:

The Cuntz-Krieger Uniqueness Theorem was proved in [2, Theorem 3.1] for row-finite graphs and in [25, Theorem 2] for arbitrary graphs. The Gauge-Invariant Uniqueness Theorem was proved in [2, Theorem 2.1] for row-finite graphs and [1, Theorem 2.1] for arbitrary graphs.

An *exit* for a cycle $\alpha_1 \cdots \alpha_n$ is an edge $f \in E^1$ such that $s(f) = s(\alpha_i)$ for some $i \in \{1, 2, \ldots, n\}$ but $f \neq \alpha_i$.

Theorem 2.1.1. (Cuntz-Krieger Uniqueness) *Let $E = (E^0, E^1, r, s)$ be a directed graph in which every cycle has an exit, and suppose that $\{s_e, p_v\}$ and $\{t_e, q_v\}$ are two Cuntz-Krieger E-families in which all the projections p_v and q_v are nonzero. Then, there is an isomorphism $\phi : C^*(\{s_e, p_v\}) \to C^*(\{t_e, q_v\})$ such that $\phi(s_e) = t_e$ and $\phi(p_v) = q_e$ for all $e \in E^1$ and $v \in E^0$.*

Theorem 2.1.2. (**Gauge-Invariant Uniqueness**) *Let $E = (E^0, E^1, r, s)$ be a directed graph, $\{S_e, P_v\}$ be a Cuntz-Krieger E-family in $B(\mathcal{H})$, and $\pi : C^*(E) \to B(\mathcal{H})$ be the representation such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. If each P_v is nonzero and there is a strongly continuous action β of $\mathbb{T}$ on $C^*(\{S_e, P_v\})$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$, then π is faithful.*

2.1.3 Strongly Connected Graphs and their Dual Graphs

The following information regarding dual graphs can be found in [46, p.17–18]:

Let $E = (E^0, E^1, r, s)$ be a graph. Define the dual graph $\widehat{E}$ by $\widehat{E}^0 = E^1$ and $\widehat{E}^1 = E^2$, where $r_{\widehat{E}}(ef) = f$ and $s_{\widehat{E}}(ef) = e$. We note that if E is row-finite, then so is $\widehat{E}$. The vertex matrix of the dual graph corresponds to the edge matrix of the original graph:

$$A_{\widehat{E}}(e, f) = \begin{cases} 1 & \text{if } ef \text{ is a path} \\ 0 & \text{if } ef \text{ is not a path} \end{cases}$$

$$= B_E(e, f)$$

We say a non-empty graph E is *strongly connected* if, for every pair of vertices $v, w \in E^0$, there is a path $|\mu| \geq 1$ such that $s(\mu) = v$ and $r(\mu) = w$.

The following proposition is straightforward and seems to be general knowledge; however, it is easy enough to provide a quick proof:

Proposition 2.1.2. *If E is a strongly connected directed graph, then so is $\widehat{E}$.*

Proof. Suppose E is strongly connected and $e, f \in \widehat{E}^0$. Let $r(e) = x$ and $s(f) = y$. Since E is strongly connected, there is a path $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ from x to y. Thus, $e\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_n f$ are paths of length two in E, so they correspond to edges in $\widehat{E}$. Hence, we have that $(e\alpha_1)(\alpha_1\alpha_2)\cdots(\alpha_{n-1}\alpha_n)(\alpha_n f)$ is a path from e to f in $\widehat{E}$, as required. □

2.1.4 Perron-Frobenius Theorem

The propositions and theorems of this section will be helpful for showing the existence of KMS states of a C^*-dynamical system (Chapter 3). It will also be useful in describing the K -theory of certain graph algebras that are used in the inductive limit construction of crossed products of graph algebras by quasi-free actions (Chapter 4).

An $m \times n$ matrix A is nonnegative, denoted $A \geq 0$, if each $a_{ij} \geq 0$. In general, $A \leq B$ if each $a_{ij} \leq b_{ij}$. Similarly, A is a positive matrix, denoted $A > 0$, when each $a_{ij} > 0$. Also, $A < B$ means that each $a_{ij} < b_{ij}$.

An $n \times n$ matrix is called a *reducible matrix* when there exists a permutation matrix P such that

$$P^{tr}AP = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

where X and Z are both square matrices. Otherwise, A is said to be an *irreducible matrix*

[42, p.671]. The following proposition will be useful in characterizing irreducible matrices:

Proposition 2.1.3. [42, p. 671] *An $n \times n$ matrix A is irreducible if and only if the corresponding graph having vertex matrix A is strongly connected.*

Theorem 2.1.3. [42, Perron-Frobenius Theorem on p.673] *Let $A \geq 0$ be an $n \times n$ irreducible matrix and let $r = \rho(A)$ denote the spectral radius of A. Then, the following are true.*

(i) $r = \rho(A)$ *is an eigenvalue of* A *and* $r > 0$.

(ii) There is a vector $\mathbf{p} > 0$ *with* $||\mathbf{p}||_1 = 1$ *such that* $A\mathbf{p} = r\mathbf{p}$. *The vector* $\mathbf{p}$ *is called the unimodular Perron-Frobenius eigenvector. There are no other nonnegative eigenvectors for* A *except for positive multiples of* $\mathbf{p}$*, regardless of the eigenvalues.*

(iii) Collatz-Wielandt formula: $r = \max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x})$, *where*

$$f(\mathbf{x}) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[A\mathbf{x}]_i}{x_i} \text{ and } \mathcal{N} = \{\mathbf{x} : \mathbf{x} \geq 0 \text{ with } \mathbf{x} \neq 0\}.$$

Remark 2.1.2. There is a "min-max" version of the Collatz-Wielandt formula: If $A \geq 0$ is an irreducible matrix and $r = \rho(A)$, then $r = \min_{\mathbf{x} \in \mathcal{P}} g(\mathbf{x})$, where

$$g(\mathbf{x}) = \max_{1 \leq i \leq n} \frac{[A\mathbf{x}]_i}{x_i} \text{ and } \mathcal{P} = \{\mathbf{x} : \mathbf{x} > 0\}.$$

The proof of this is similar to the "max-min" version of the Collatz-Wielandt formula for positive matrices (see the bottom of page of 666 in [42]). The "min-max" version of the Collatz-Wielandt formula is used in Proposition 3.4.2.

Even if A is not irreducible, it is still possible to find an eigenvector that has a corresponding eigenvalue equal to the spectral radius $\rho(A)$ of A. This is described in the next proposition.

Proposition 2.1.4. [42, p. 670] *Let A be an $n \times n$ matrix with $A \geq 0$ and let $r = \rho(A)$ denote the spectral radius of A. Then the following are true:*

(i) $r = \rho(A)$ *is an eigenvalue of* A *(*$r = 0$ *is possible);*

(ii) $A\mathbf{z} = r\mathbf{z}$ *for some* $\mathbf{z} \in \mathcal{N} = \{\mathbf{x} : \mathbf{x} \geq 0 \text{ with } \mathbf{x} \neq 0\}$*;*

(iii) $r = \max_{z \in \mathcal{N}} f(\mathbf{x})$*, where*

$$f(\mathbf{x}) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[A\mathbf{x}]_i}{x_i} \text{ and } \mathcal{N} = \{\mathbf{x} : \mathbf{x} \geq 0 \text{ with } \mathbf{x} \neq 0\}.$$

2.1.5 Characterization of Extreme Points for Polyhedral Sets

For the C^*-dynamical systems discussed in this thesis, the simplex of KMS states is shown to be affine isomorphic to a polyhedral set of the form $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. (Theorem 3.3.1). The characterization of the extreme points of this polyhedral set S is described below and will be useful when computing the KMS states for specific C^*-dynamical systems (Section 3.9).

Definition 2.1.3. A set S in $\mathbb{R}^n$ is called a *polyhedral set* if it is the intersection of a finite number of closed half-spaces; that is, $S = \{\mathbf{x} : \mathbf{p}_i^t\mathbf{x} \leq \alpha_i : \text{ for } i = 1, \ldots, m\}$, where $\mathbf{p}_i$ is a nonzero vector and α_i is a scalar for $i = 1, \ldots, m$.

The theorem below gives a characterization of the extreme points of the polyhedral set $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where A is an $m \times n$ matrix of rank m and $\mathbf{b}$ is a vector in $\mathbb{R}^m$.

We can rewrite $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \geq \mathbf{0}$ as follows: rearrange the columns of A and write $A = [B|N]$, where B is some $m \times m$ matrix of full row rank and N is an

$m \times (n-m)$ matrix and $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$. Then, $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ is equivalent to

$$B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b} \text{ and } \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0}.$$

$\mathbf{x}$ is called a basic feasible solution for S if $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{x}_B = B^{-1}\mathbf{b}$.

Theorem 2.1.4 (Theorem 2.6.4 [3, pp. 67– 69])**.** *Let $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where A is an $m \times n$ matrix of rank m and $\mathbf{b}$ is a vector in $\mathbb{R}^m$. A point $\mathbf{x}$ is an extreme point of S if and only if $\mathbf{x}$ is a basic feasible solution for S.*

2.1.6 K-Theory

In Chapter 4, the K-theory of certain graph algebras is computed. K-theory associates to each C^*-algebra A, two abelian groups $K_0(A)$ and $K_1(A)$ that contains information about A. The group $K_0(A)$ is defined via a Grothendieck construction from an abelian semigroup. When A is unital, it can be viewed as the set of formal differences of Murrary-von Neumann equivalence classes projections. There is also a general definition of $K_0(A)$ for the case when A is nonunital ([50, pp. 55–70]). Here are a few key facts that will be used in this thesis:

- **K_0 group of finite dimensional C^*-algebras** ([50, p. 46-47, 67]):

  $$K_0(M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})) = \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \oplus \cdots \oplus \mathbb{Z}^{n_k}.$$

- **Continuity of K_0**([50, pp. 98–99]): For each inductive sequence

  $$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots$$

  of C^*-algebras, $K_0(\varinjlim A_n)$ and $\varinjlim K_0(A_n)$ are isomorphic as abelian groups.

$K_1(A)$ is defined in terms of equivalence classes of unitaries ([50, p.134]). For this thesis, the K-theory that is calculated is for AF algebras and it is known that if A is an AF algebra, $K_1(A) = 0$ ([50, p.235]).

For graph algebras, the K-theory can be computed in the following way:

Theorem 2.1.5. [46, Theorem 7.16] *Let E be a row-finite directed graph without sinks and let A_E be the vertex matrix of E. Then, $K_1(C^*(E))$ is isomorphic to the kernel of $1 - A_E^{tr} : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$, and $K_0(C^*(E))$ is isomorphic to the cokernel.*

2.2 KMS States and Ground States

The definitions and ideas described in this section can be found in [5].

In the C^*-algebraic formulation of a quantum statistical system, the self-adjoint elements of a C^*-algebra A are the *observables* of the system, the *states* of the system are the positive linear functionals of norm 1, and the *expected value* of the observable a in a state ϕ is $\phi(a)$. An action $\alpha : \mathbb{R} \to \mathrm{Aut}A$ is known as the *time evolution* of the system and it moves the expected value $\phi(a)$ of an observable at time 0 to $\phi(\alpha_t(a))$, when the system is in the state ϕ. If a system is in *equilibrium*, the expectation value of each of its observables is constant in time. The KMS states, which we define below, are exactly the equilibrium states for finite-dimensional systems but make sense in the context of infinite-dimensional systems as well.

Given a C^*-algebra A and a homomorphism $\sigma : \mathbb{R} \to \mathrm{Aut}A$, an element $a \in A$ is called *analytic* if $t \mapsto \sigma_t(a)$ extends to an entire function on $\mathbb{C}$. For $\beta \in (0, \infty)$, a KMS state of (A, σ) at *inverse temperature* β is a state ϕ of A that satisfies the KMS_β condition

$$\phi(ab) = \phi(b\sigma_{i\beta}(a)) \tag{2.1}$$

for all a, b analytic in A (as in[5]). A KMS_0 state of (A, σ) is a state ϕ of A that is invariant, with respect to σ, and that satisfies the trace condition $\phi(ab) = \phi(ba)$ for all $a, b \in A$ (as in [5]). A KMS_∞ state is a weak* limit of a sequence of KMS_{β_n} states, as $\beta_n \to \infty$, and a ground state is a state ϕ such that the functions $\phi_{a,b} : z \mapsto \phi(a\alpha_z(b))$ are bounded in the upper-half plane for every a, b analytic in A (as in [10]).

2.3 Crossed Products

2.3.1 C^*-dynamical Systems

The definitions and examples in this section can be found in [44] and [57].

Definition 2.3.1. An action of a locally compact group G on a C^*-algebra A is a homomorphism $g \mapsto \alpha_g$ of G into the group $\text{Aut}A$ of automorphisms of A. An action is a strongly continuous action if $g \mapsto \alpha_g(a)$ is continuous for each fixed $a \in A$. Throughout this thesis, all of the actions that are considered will be strongly continuous.

Definition 2.3.2. A C^*-dynamical system, denoted by (A, G, α), consists of a C^*-algebra A, a locally compact group G and a strongly continuous action α of G on A.

Example 2.3.1. Let G be locally compact group and let X be a locally compact Hausdorff space. Then $(C_0(X), G, \alpha)$ is a dynamical system, where an action $(g, x) \mapsto g \cdot x$ of G on X by homeomorphisms induces an action of G on $C_0(X)$, given by

$$\alpha_g(f)(x) = f(g^{-1} \cdot x).$$

Definition 2.3.3. Let (A, G, α) be a dynamical system. Then, a covariant representation of (A, G, α) is a pair (π, U) consisting of a non-degenerate representation $\pi : A \to B(\mathcal{H})$ and a unitary representation $U : G \to U(\mathcal{H})$ on the same Hilbert space such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*.$$

Definition 2.3.4. (Crossed Product) Define a multiplication (convolution) and involution on $C_c(G, A)$ by

$$(f * g)(t) = \int_G f(s)\alpha_s(g(s^{-1}t))ds$$

and

$$f^*(t) = \Delta_G(t^{-1})\alpha_t(f(t^{-1})^*),$$

where ds denotes the Haar measure and Δ denotes the modular function for the group. Then, with $||f||_1 := \int_G ||f(t)||dt$, $C_c(G, A)$ becomes a $*$-normed algebra. Let $L^1(G, A)$ be the completion.

Given a covariant representation (π, U) of (A, G, α) on $\mathcal{H}$, define a representation $\pi \rtimes U$ of $L^1(G, A)$ on $\mathcal{H}$ by

$$(\pi \rtimes U)(f) = \int_G \pi(f(s))U_s ds.$$

Define a norm on $L^1(G, A)$ by

$$||f|| := \sup\{||(\pi \rtimes U)(f)|| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

Then, the completion of $L^1(G, A)$, with respect to this norm, is a C^* algebra called the *crossed product* of (A, G, α) and is denoted by $A \rtimes_\alpha G$.

Remark 2.3.1. It is worth noting that different actions can produce isomorphic crossed products. If θ is an automorphism of G and $\beta_t = \alpha_{\theta(t)}$ for all $t \in G$, then $A \rtimes_\beta G$ is isomorphic to $A \rtimes_\alpha G$ ([4, pp.218–219]). This fact will be used in Chapter 4 when certain quasi-free actions are scaled to produce isomorphic crossed products.

2.4 Quasi-Free Actions

Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family that generates $C^*(E)$. Then, there is a strongly continuous action $\gamma : \mathbb{T} \to \mathrm{Aut}C^*(E)$ such that $\gamma_z(s_e) = zs_e$ for all $e \in E^1$ and $\gamma_z(p_v) = p_v$ for all $v \in E^0$ ([46, Proposition 2.1]). This action is known as the **gauge action**. Proposition 2.4.1 below defines a class of new action of the reals that are known as the **quasi-free actions** or the generalized gauge actions. The quasi-free action α^ω of $\mathbb{R}$ on $C^*(E)$ corresponds to a labeling map ω on E^1. This labeling map has an extension to E^*, which we also denote by ω, and is defined below. For the case where $\omega \equiv 1$, the quasi-free action α^ω will be referred to as the gauge action of the reals.

Definition 2.4.1. A map $\omega : E^1 \to \mathbb{R}$ is called a labeling map on E^1. We extend ω to E^* by $\omega(\mu) = \omega(\mu_1) + \cdots + \omega(\mu_n)$ for $\mu = \mu_1 \cdots \mu_n \in E^* \backslash E^0$ and $\omega(v) = 0$ for $v \in E^0$.

In [21], Evans defined quasi-free actions on the Cuntz-algebra O_n. An extension to general graph algebras is well known and taken as general knowledge. A proof of Proposition 2.4.1 can be found in [14].

Proposition 2.4.1. *Let $\omega : E^1 \to \mathbb{R}$ be a labeling map on E^1. Then, there is a strongly continuous action $\alpha^\omega : \mathbb{R} \to \mathrm{Aut}\, C^*(E)$ such that $\alpha_t(s_e) = e^{i\omega(e)t}s_e$ for all $e \in E^1$ and $\alpha_t(p_v) = p_v$ for all $v \in E^0$.*

Remark 2.4.1. We note that the labeling that is used in [14] is a map $c : E^* \to \mathbb{R}_+^*$, defined by $c(v) = 1$ if $v \in E^0$ and $c(\mu) = c(\mu_1)c(\mu_2)\cdots c(\mu_n)$ if $\mu = \mu_1\mu_2\cdots\mu_n \in E^n$. We use the labeling $\omega(v) = 0$ for all vertices v and $\omega(\mu) = \omega(\mu_1) + \cdots + \omega(\mu_n)$.

Chapter 3

KMS States for Quasi-Free Actions on Finite-Graph Algebras

The main results of this thesis deal with extending the results from the paper, *KMS states on the C^*-algebras of finite graphs* by an Huef, Laca, Raeburn, and Sims [29]. Their paper analyzed the KMS states of the C^*-dynamical systems consisting of the Toeplitz algebra of a graph along with the gauge action of the reals. The results obtained in this chapter extend the results in [29] to quasi-free actions.

In Section 3.1, the Toeplitz algebra is defined and the main results from [29] are stated. Section 3.2 states some useful preliminaries that are needed in extending the results in [29] to quasi-free actions. The main result of Section 3.3 is Theorem 3.3.1, which characterizes the simplex of all KMS states in terms of a polyhedral set in $\mathbb{R}^{E^0}$. This theorem is very useful in extending the results of [29] and in addition, allows one to readily compute the KMS states for a given quasi-free action on a finite graph algebra. We illustrate this through examples in Section 3.9. The main results of Section 3.4 deal with showing the existence of a KMS state at a critical inverse temperature β_c. In Section 3.5, a precise description of the KMS states above a critical inverse temperature is given, extending Theorem 3.1 of [29]. Section 3.6

looks at graphs with strongly connected subgraphs, analyzes their KMS states, and gives a complete description of the KMS states in this case (Theorem 3.6.1). As a consequence, we extend Theorem 4.3 in [29]. Section 3.7 investigates the connection between the KMS states of a graph algebra and its dual-graph algebra, when the graph is row-finite and has no sinks. Section 3.8 extends Proposition 5.1 of [29].

3.1 KMS States for the Gauge Action on Finite-Graph Algebras

In this section, we discuss the definition of the Toeplitz algebra, as well as the main results from [29], which will be extended to quasi-free actions below.

3.1.1 Toeplitz Algebra

The definition of the Toeplitz algebra $\mathcal{T}C^*(E)$ is similar to that of the graph algebra. Suppose E is a directed graph. A Toeplitz-Cuntz-Krieger E-family $\{p_v, s_e\}$ consists of a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a collection of partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges such that

- $s_e^* s_e = p_{r(e)}$
- $p_v \geq \sum_{e \in F} s_e s_e^*$ for every finite subset F of vE^1.

The Toeplitz algebra $\mathcal{T}C^*(E)$ is the C^*-algebra generated by a universal Toeplitz-Cuntz-Krieger family (the existence is justified in [26]). Also, If $\{p_v, s_e\}$ is a universal Toeplitz-Cuntz-Krieger family, then

$$\mathcal{T}C^*(E) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\},$$

by arguments similar to those used for graph algebras [46].

The graph algebra $C^*(E)$ is the quotient of the Toeplitz algebra $\mathcal{T}C^*(E)$ by the ideal J generated by $P =: \{p_v - \sum_{f\in vE^1} s_f s_f^* : v \in E^0 \text{ and } vE^1 \neq \emptyset\}$.

As with graph algebras, the Toeplitz algebra carries the gauge action of $\mathbb{R}$ which satisfies $\gamma_t(s_\mu s_\nu^*) = e^{it(|\mu|-|\nu|)} s_\mu s_\nu^*$. The theorem below characterizes the simplex of KMS_β states of $(\mathcal{T}C^*(E), \gamma)$ for $\beta > \ln\rho(A)$, where $\gamma : \mathbb{R} \to \text{Aut}\mathcal{T}C^*(E)$ is the gauge action of the reals.

Theorem 3.1.1. [29, Theorem 3.1] *Let E be a finite directed graph with vertex matrix $A \in M_{E^0}(\mathbb{N})$. Let $\gamma : \mathbb{R} \to \text{Aut}\mathcal{T}C^*(E)$ be the gauge action of the reals. Fix a vector $y = (y_v)_{v\in E^0}$ in $\mathbb{R}^{E^0}$ by $y_v = \sum_{\mu\in E^* v} e^{-\beta|\mu|}$ and assume that $\beta > \ln\rho(A)$. Then,*

$$\textstyle\sum_\beta := \{\epsilon \in [0,\infty)^{E^0} : \epsilon \cdot y = 1\}$$

is affine isomorphic to the simplex of KMS$_\beta$ states of $(\mathcal{T}C^*(E), \gamma)$ via a map which sends $\epsilon \longmapsto \phi_\epsilon$, where $\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} m_{r(\mu)}$ and $m := (I - e^{-\beta}A)^{-1}\epsilon$. The inverse of this isomorphism takes the KMS$_\beta$ state ϕ to $(I - e^{-\beta}A)m^\phi$, where $m^\phi := (\phi(p_v))_v$.

3.1.2 KMS States at a Critical Inverse Temperature

In [29, Proposition 4.1], it was shown that the existence of a probability measure m on E^0 that satisfies $Am \leq \rho(A)m$ gives rise to a $\text{KMS}_{\ln\rho(A)}$ state ϕ on $(\mathcal{T}C^*(E), \gamma)$ such that

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu}\rho(A)^{-|\mu|} m_{r(\mu)}.$$

As a consequence, $(\mathcal{T}C^*(E), \gamma)$ has a $\text{KMS}_{\ln\rho(A)}$ state when E is a finite graph [29, Corollary 4.2].

3.1.3 Strongly Connected Graphs

For a strongly connected finite graph E, a complete description of the KMS states of the system $(\mathcal{T}C^*(E), \gamma)$ is now given.

(a) If $\beta > \ln \rho(A)$, then the simplex of KMS$_\beta$ states of $(\mathcal{T}C^*(E), \gamma)$ is affine-isomorphic to

$$\textstyle\sum_\beta := \{\epsilon \in [0, \infty)^{E^0} : \epsilon \cdot y = 1\}.$$

The KMS states are described in Theorem 3.1.1.

(b) [29, Theorem 4.3]The system $(\mathcal{T}C^*(E), \gamma)$ has a unique KMS$_{\ln \rho(A)}$ state ϕ. This state satisfies

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \rho(A)^{-|\mu|} m_{r(\mu)}. \tag{3.1}$$

The state factors through a KMS$_{\ln \rho(A)}$ state $\bar{\phi}$ of $(C^*(E), \gamma)$. In addition, the state $\bar{\phi}$ is the only KMS state of $(C^*(E), \gamma)$.

(c) [29, Theorem 4.3]If $\beta < \ln \rho(A)$, then $(\mathcal{T}C^*(E), \gamma)$ has no KMS$_\beta$ states.

3.1.4 Ground States

The result in [29, Proposition 5.1] described the ground states of $(\mathcal{T}C^*(E), \gamma)$.

Proposition 3.1.1. [29, Proposition 5.1] *Let E be a finite directed graph and let $\gamma : \mathbb{R} \to \mathrm{Aut}\mathcal{T}C^*(E)$ be the gauge action of $\mathbb{R}$. Suppose that ϵ is a probability measure on E^0. Then, there is a KMS$_\infty$ state ϕ_ϵ satisfying*

$$\phi_\epsilon(s_\mu s_\nu^*) = \begin{cases} 0 & \textit{unless } |\mu| = |\nu| = 0 \textit{ and } \mu = \nu \\ \epsilon_v & \textit{if } \mu = \nu = v \in E^0. \end{cases}$$

Every ground state of $(\mathcal{T}C^*(E), \gamma)$ is a KMS$_\infty$ state and the map $\epsilon \longmapsto \phi_\epsilon$ is an affine isomorphism of the simplex of probability measures of E^0 onto the set of ground states of $(\mathcal{T}C^*(E), \gamma)$.

3.1.5 Graphs with Sinks

The role of sinks in a graph is explored in the Proposition below. A KMS state of $(C^*(E), \gamma)$, where $\gamma : \mathbb{R} \to \mathrm{Aut}C^*(E)$ is the gauge action of the reals, can be described when a certain subgraph is strongly connected.

Proposition 3.1.2. [29, Corollary 6.1] *Let E be a finite directed graph with vertex matrix A and suppose that E has no sources. Let H be the saturation of the set of sinks in E and suppose that the graph $E\backslash H := (E^0\backslash H, E^1\backslash r^{-1}(H), r, s)$ is strongly connected. Then there is a unique KMS$_{\ln \rho(A)}$ state ϕ of $(C^*(E), \gamma)$ and*

$$\phi_\epsilon(s_\mu s_\nu^*) = \begin{cases} \delta_{\mu,\nu}\rho(A)^{-|\mu|}x_{r(\mu)} & \text{if } r(\mu) \in E^0\backslash H \\ 0 & \text{if } r(\mu) \in H, \end{cases}$$

where x is the unimodular Perron-Frobenius eigenvector for the vector matrix $A_{E\backslash H}$ of $E\backslash H$.

Remark 3.1.1. Most of the results in [29] involve the Toeplitz algebra $\mathcal{T}C^*(E)$ and the gauge action of the reals $\gamma : \mathbb{R} \to \mathcal{T}C^*(E)$, which satisfies $\gamma_t(s_\mu s_\nu^*) = e^{it(|\mu|-|\nu|)}s_\mu s_\nu^*$. Given a labeling map $\omega : E^1 \to \mathbb{R}$, the Toeplitz algebra also carries a quasi-free action given by $\alpha_t^\omega(s_\mu s_\nu^*) = e^{it(\omega(\mu)-\omega(\nu))}s_\mu s_\nu^*$. In this thesis, we view the Toeplitz algebra as a graph algebra by relying on a result that is proved in [43]. The graph is constructed in Definition 3.2.1 and the resulting isomorphism between the algebras is stated in Proposition 3.2.1.

3.2 Preliminaries

This section contains some important preliminary definitions and results that are needed to extend the results described in Section 3.1.

3.2.1 Expressing the Toeplitz Algebra as a Graph Algebra

The Toeplitz algebra $\mathcal{T}C^*(E)$ is isomorphic to the graph algebra $C^*(E_{\mathcal{T}})$, where the associated graph $E_{\mathcal{T}}$ comes from E and is defined below.

Definition 3.2.1. [43, Definition 3.6] Let $E = (E^0, E^1, r, s)$ be a graph and recall $E^0_{\text{reg}} := \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$. Define a new graph $E_{\mathcal{T}}$ by letting

$$E^0_{\mathcal{T}} := E^0 \cup \{v' : v \in E^0_{\text{reg}}\}$$
$$E^1_{\mathcal{T}} := E^1 \cup \{e' : e \in E^1 \text{ and } r(e) \in E^0_{\text{reg}}\}$$

with range and source maps extended to $E^1_{\mathcal{T}}$ by $s(e') = s(e)$, and $r(e') = r(e)'$.

Proposition 3.2.1. [43, Theorem 3.7] *Let E be a graph and let $\{s_e, p_v\}$ be a generating Toeplitz-Cuntz-Krieger E-family in $\mathcal{T}C^*(E)$. Then, the Toeplitz algebra $\mathcal{T}C^*(E)$ is canonically isomorphic to the graph algebra $C^*(E_{\mathcal{T}})$. Furthermore, if we define*

$$q_w := \begin{cases} p_w & \text{if } w \notin E^0_{reg} \\ \sum_{\{e \in E^1 : s(e) = w\}} s_e s_e^* & \text{if } w \in E^0_{reg} \\ p_v - \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^* & \text{if } w = v' \text{ for some } v \in E^0_{reg} \end{cases}$$

$$t_f := \begin{cases} s_f q_{r(f)} & \text{if } f \in E^1 \\ s_e q_{r(e)'} & \text{if } f = e' \text{ for some } e \in E^1 \end{cases}$$

then $\{t_f, q_w\}$ generates a Cuntz-Krieger $E_{\mathcal{T}}$-family in $\mathcal{T}C^*(E)$.

Remark 3.2.1. The above proposition is just a specific example of a more general result: it was shown that every relative graph algebra $C^*(E, V)$, where $V \subseteq R(E)$, is canonically isomorphic to the graph algebra $C^*(E_V)$ (see [43]). Since the Toeplitz algebra $\mathcal{T}C^*(E)$ is the relative graph algebra $C^*(E, \emptyset)$, we adopted the notation $C^*(E_{\mathcal{T}})$ instead of $C^*(E_\emptyset)$.

Proposition 3.2.2. *Let $E = (E^0, E^1, r, s)$ be a graph. Let $\omega : E^1 \to \mathbb{R}$ be a labeling map on E^1 and extend ω to $E^1_{\mathcal{T}}$ by $\omega(e') = \omega(e)$. Then, $(C^*(E_{\mathcal{T}}), \alpha^\omega)$ is covariantly isomorphic to $(\mathcal{T}C^*(E), \alpha^\omega)$.*

Proof. Follows immediately from Proposition 3.2.1. □

3.2.2 KMS States and Tracial States

In [14, Theorem 3.3], it was shown that if $\omega(\mu) \neq 0$ for all $\mu \in E^* \backslash E^0$, then a state σ is a KMS$_\beta$ state of $(C^*(E), \alpha^\omega)$ if and only if

$$\sigma(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} \sigma(p_{r(\mu)}). \tag{3.2}$$

In [14, Theorem 3.10], it was shown that there is a bijective correspondence between the KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$ and a certain class of tracial states on $C_0(E^0)$:

Definition 3.2.2. [14, Definition 3.4] Let ω be a labeling map on E^1 that is bounded below and let $\beta \geq 0$. Given a tracial state τ on $C_0(E^0) \cong \overline{\text{span}}\{p_v\}_{v \in E^0}$, we can define a trace on $C_0(E^0)$ by

$$\mathcal{F}_{\omega,\beta}(\tau)(p_v) = \lim_{D \to s^{-1}(v)} \sum_{e \in D} e^{-\beta\omega(e)} \tau(p_{r(e)}),$$

where the limit is taken on finite subsets D of $s^{-1}(v)$ and $\mathcal{F}_{\lambda,\beta}(\tau)(p_v) = 0$ if $s^{-1}(v) = \emptyset$ [14].

Theorem 3.2.1. [14, Theorem 3.10] *Let γ be the standard gauge action of $\mathbb{T}$ on $C^*(E)$ and $C^*(E)^\gamma$ the fixed-point subalgebra of $C^*(E)$. Let ω be a labeling map on E^1 that is bounded below and let $\beta \geq 0$. If σ is a state on $C^*(E)^\gamma$ satisfying (3.2), then its restriction τ to $C_0(E^0)$ satisfies:*

(K1) $\mathcal{F}_{\omega,\beta}(\tau)(a) = \tau(a)$ *for all* $a \in \overline{\text{span}}\{p_v : 0 < |s^{-1}(v)| < \infty\}$

(K2) $\mathcal{F}_{\omega,\beta}(\tau)(a) \leq \tau(a)$ *for all* $a \in C_0(E^0)^+$.

Conversely, if τ is a tracial state on $C_0(E^0)$ satisfying $(K1)$ and $(K2)$, then there is a unique state σ on $C^*(E)^\gamma$ satisfying (3.2) with $\sigma|_{C_0(E^0)} = \tau$.

If γ is the gauge action of $\mathbb{T}$ on $C^*(E)$, then the fixed-point subalgebra $C^*(E)^\gamma$ is called the *core* of $C^*(E)$ and we will use this terminology throughout the thesis.

3.3 Characterizing KMS States for Quasi-Free Actions

In this section, we characterize the KMS$_\beta$ states of $C^*(E)$ in terms of vectors that satisfy a certain *Property* P_β defined below. When E is a finite graph (that is, E^0 and E^1 are both finite sets), the simplex of all KMS$_\beta$ states of $C^*(E)$ can be viewed as a polyhedral set in $\mathbb{R}^{E^0}$, and, in turn, we can readily compute its extreme points.

Definition 3.3.1. Let $E = (E^0, E^1, r, s)$ be a row-finite graph and $\alpha^\omega : \mathbb{R} \curvearrowright C^*(E)$ be the quasi-free action that corresponds to the labeling map $\omega : E^1 \to \mathbb{R}$. Let $\beta \in \mathbb{R}$ and $C_\beta \in M_{E^0}(\mathbb{R})$ the matrix defined by $C_\beta(v, w) := \sum_{e \in vE^1 w} e^{-\beta\omega(e)}$. Note that, if $vE^1w = \emptyset$, then $C_\beta(v, w) = 0$, by standard convention. (The matrix C_β may also be written as $C_{\beta,E}$ or $C_{\beta,E,\omega}$, if we need to be more specific). We say that a vector $m := (m_v)_{v\in E^0}$ satisfies *Property* P_β on E^0 if m is a probability measure with $(C_\beta m)_v = m_v$, whenever v is a regular vertex.

Remark 3.3.1. If we reduce to the gauge action, we note that $C_\beta = e^{-\beta}A$, where A is the vertex matrix of E. Note that, if m satisfies Property P_β on E^0, then m satisfies the subinvariance relation $C_\beta m \leq m$. That is, $(C_\beta m)_v \leq m_v$ for each $v \in E^0$. Also, if E is a strongly connected graph, then C_β is an irreducible matrix.

Theorem 3.3.1. *Let E be a row-finite graph, ω a labeling map on E^1 that is bounded below, and $\beta \geq 0$. Let K_{β,α^ω} be the set of all KMS$_\beta$ states for the quasi-free action α^ω on $C^*(E)$ and let $L_{\beta,\alpha^\omega} := \{m = (m_v)_{v\in E^0} : m$ satisfies Property P_β on $E^0\}$. Suppose that ω satisfies $\omega(\mu) \neq 0$ for all $\mu \in E^*\backslash E^0$. Then, K_{β,α^ω} is affine-isomorphic to L_{β,α^ω}. More specifically, for each $m \in L_{\beta,\alpha^\omega}$, the corresponding KMS$_\beta$ state ϕ_m satisfies*

$$\phi_m(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} m_{r(\mu)}. \tag{3.3}$$

Proof. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family that generates $C^*(E)$. Let $\pi : C^*(E) \to B(\mathcal{H})$ be a faithful nondegenerate representation and define $P_v := \pi(p_v)$ and $S_e := \pi(s_e)$. Define a map $\Psi : K_{\beta,\gamma} \to \mathbb{R}^{E^0}$ by $\Psi(\phi) = m^\phi$, where $m^\phi := (\phi(p_v))_{v\in E^0}$. Clearly, Ψ is an affine map that is weak*-continuous. Since ϕ is a state, we have that m^ϕ is a probability measure. Also, whenever v is a regular vertex, we have

$$\begin{aligned} m_v^\phi = \phi(p_v) &= \sum_{e\in vE^1} \phi(s_e s_e^*) = \sum_{e\in vE^1} e^{-\beta\omega(e)}\phi(p_{r(e)}) \\ &= \sum_{w\in E^0}\sum_{e\in vE^1 w} e^{-\beta\omega(e)}\phi(p_w) = \sum_{w\in E^0} C_\beta(v,w)\phi(p_w) \\ &= (C_\beta m^\phi)_v. \end{aligned}$$

Thus, $(C_\beta m^\phi)_v = m_v^\phi$ and $m^\phi \in L_{\beta,\alpha^\omega}$.

To show the image of Ψ is L_{β,α^ω}, choose an $x \in L_{\beta,\alpha^\omega}$. Define a tracial state τ on $C_0(E^0)$ by $\tau(a) = \sum_{v\in E^0} x_v(\pi|_{C_0(E^0)}(a)P_v, P_v)$. Indeed, $\tau(1) = \sum_{v\in E^0} x_v = 1$. If v is a regular vertex, then

$$\begin{aligned}
\mathcal{F}_{\omega,\beta}(\tau)(p_v) &= \sum_{e\in vE^1} e^{-\beta\omega(e)}\tau(p_{r(e)}) = \sum_{w\in E^0}\sum_{e\in vE^1w} e^{-\beta\omega(e)}x_w \\
&= \sum_{w\in E^0} C_\beta(v,w)x_w \qquad (3.4)\\
&= x_v \qquad (3.5)\\
&= \tau(p_v),
\end{aligned}$$

where (3.4) equals (3.5) since $x \in L_{\beta,\alpha^\omega}$. By Theorem 3.2.1, we have a unique state σ on the core of $C^*(E)$ that satisfies (3.2) with $\sigma|_{C_0(E^0)} = \tau$. Hence, $\phi = \sigma \circ \Phi$ is a KMS$_\beta$ state on $C^*(E)$ by Theorem 3.3 of [14]. So, $\Psi(\phi)(v) = \phi(p_v) = \sigma(p_v) = x_v$ and $\Psi(\phi) = x$.

To prove injectivity, suppose $\Psi(\phi_1) = \Psi(\phi_2)$. Then, $\phi_1(p_v) = \phi_2(p_v)$ for all $v \in E^0$. Hence, by Proposition 3.2 of [14], both KMS$_\beta$ states coincide on its core. Since $\omega(\mu) \neq 0$ for all $\mu \in E^*$, the KMS$_\beta$ states are equal. □

Notice that the result above is analogous to [14, Definition 3.4]. The map below allows one to view L_{β,α^ω} as a polyhedral set in $\mathbb{R}^{E^0}$ (see Remark 3.3.2). In addition, when E is a finite directed graph the map gives an affine homeomorphism.

Remark 3.3.2. We note that if $x = (x_v)_{v\in E^0}$ is in L_{β,α^ω}, it will satisfy the following equations:

$$x_v - \sum_{w\in E^0} C_\beta(v,w)x_w = 0 \text{ for each } v \in E^0_{\text{reg}}$$

$$\sum_{w\in E^0} x_w = 1.$$

Let R_β be the coefficient matrix of the linear system above and $d = (0\ 0 \cdots 1)^{tr}$.

Then, we have that

$$L_{\beta,\alpha^\omega} = \{x = (x_v)_{v\in E^0} : R_\beta x = d, x \geq 0\}. \tag{3.6}$$

For the gauge action γ of $\mathbb{R}$, we can multiply each row (except the row of ones) of R_β by e^β to allow for simpler calculations (see Example 3.9.1). We denote this matrix by $\tilde{R}_\beta$.

When E is a finite graph, L_{β,α^ω} is a polyhedral set. Thus, we can easily calculate the extreme points of the set K_{β,α^ω} of all KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$ (see the examples in Section 3.9 below).

3.4 KMS States at a Critical Inverse Temperature

In this section, we show that there exists a KMS state at the critical inverse temperature $\beta_c \geq 0$, where β_c satisfies $\rho(C_{\beta_c}) = 1$ and C_{β_c} is the matrix defined in Definition 3.3.1. First, we will prove the existence and uniqueness of β_c. We recall that the edge matrix of E is the matrix $B \in M_{E^1}(\mathbb{N})$ defined by

$$B(e,f) = \begin{cases} 1 & \text{if } ef \text{ is a path} \\ 0 & \text{if } ef \text{ is not a path.} \end{cases}$$

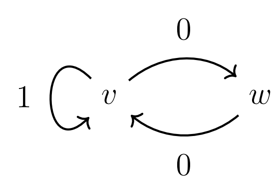
Lemma 3.4.1. *Let E be a strongly connected finite graph with $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$). For each β, let D_β be the diagonal matrix in $M_{E^1}(\mathbb{R})$ having diagonal entries $e^{-\beta\omega(e)}$ and $B \in M_{E^1}(\mathbb{N})$ the edge matrix of E. Then, there exists a unique $\beta_c \geq 0$ ($\beta_c \leq 0$) with $\rho(D_{\beta_c}B) = 1$. Furthermore, if E is a strongly connected graph that consists of a single cycle, then $\beta_c = 0$. Otherwise, if $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$), then $\beta_c > 0$ ($\beta_c < 0$).*

Proof. Assume $\omega(e) > 0$ for all $e \in E^1$. Since E is strongly connected, so is $\widehat{E}$ by

Proposition 2.1.2. Thus, B is irreducible and this implies that $D_\beta B$ is irreducible for each β. By Proposition 18.3 of [22], there exists a β_c that satisfies $\rho(D_{\beta_c}B) = 1$. By the Perron-Frobenius Theorem, $\rho(D_{\beta_c}B) = 1$ if and only if we have an eigenvector $(y_f)_f$ with $y_f > 0$ and $\sum_{f\in E^1} y_f = 1$ such that $D_{\beta_c}By = y$. Let $a_e := \sum_{f\in E^1} B(e,f)y_f$ and $\varphi(x) = \sum_{e\in E^1} a_e x^{\omega(e)} - 1$. Since E is strongly connected, $|r^{-1}(s(e))| \geq 1$ for every $e \in E^1$ and thus $\sum_{e\in E^1} a_e = \sum_{e\in E^1} |r^{-1}(s(e))|y_e \geq 1$. Hence, φ is a real valued function that has a unique positive real root $\xi \in (0,1]$. Since $D_{\beta_c}By = y$, we have that $\xi = e^{-\beta_c}$ and therefore, $\beta_c \geq 0$.

If E is a cycle, then $|r^{-1}(s(e))| = 1$ for all $e \in E^1$ and hence $\beta_c = 0$. Otherwise, there is a vertex $v \in E^0$ that receives two edges, say e and f. Since E is strongly connected, v is not a sink and so, it emits some edge $g \in E^1$. So, $e, f \in r^{-1}(s(g))$ and we get that $|r^{-1}(s(g))| \geq 2$. Thus, $\sum_{e\in E^1} a_e > 1$ and therefore, $\beta_c > 0$. □

Remark 3.4.1. If E is strongly connected and $\omega(e) = 0$ for all $e \in E^1$, then $\rho(D_\beta B) = \rho(B) = \rho(A)$ by Proposition 4.1 in [41]. Hence, if E is a cycle, then $\rho(D_\beta B) = 1$ for all β by Lemma A.1 in [29]. Otherwise, $\rho(A) > 1$ and there is no such β. Also note that if $\omega(e) = 0$ for some, but not all edges $e \in E^1$, then there need not exist a β that satisfies $\rho(D_\beta B) = 1$. For example, let E be the graph below having labels 0 and 1.



Then, $\rho(D_\beta B) \neq 1$ for all $\beta \in \mathbb{R}$.

Proposition 3.4.1. *Let E be a strongly connected finite graph with $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$). Then, there exists a unique $\beta_c \geq 0$ ($\beta_c \leq 0$) with $\rho(C_{\beta_c}) = 1$. Furthermore, if E is a strongly connected graph that consists of a single cycle, then $\beta_c = 0$. Otherwise, if $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all*

$e \in E^1$), *then* $\beta_c > 0$ ($\beta_c < 0$).

Proof. Let S_β be the $E^0 \times E^1$ matrix defined by

$$S_\beta(v,e) = \begin{cases} e^{-\beta\omega(e)} & \text{if } s(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

and let R be the $E^1 \times E^0$ matrix defined by

$$R(e,v) = \begin{cases} 1 & \text{if } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Then, $RS_\beta = BD_\beta$ and $S_\beta R = C_\beta$. Since $\rho(C_\beta) = \rho(BD_\beta) = \rho(D_\beta B)$, we have that the rest follows from Lemma 3.4.1. □

Below, we extend Proposition 3.4.1 by making use of a *Seneta decomposition* [53, Section 1.2, pp. 11–17], which decomposes C_β as a block triangular matrix, where the diagonal blocks are either 1×1 zero matrices or are irreducible.

Proposition 3.4.2. *Let E be a finite graph with $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$). Then, there exists a unique $\beta_c \geq 0$ ($\beta_c \leq 0$) with $\rho(C_{\beta_c}) = 1$.*

Proof. Let $F_1, F_2, \ldots, F_n$ be the strongly connected components of E. From the Seneta decomposition of C_β, we have that $\rho(C_\beta) = \max\{\rho(C_{\beta,F_k}) : k = 1, 2, \ldots, n\}$, where each C_{β,F_k} is an irreducible matrix (see [53] and [28]). For each $k = 1, 2, \ldots n$, there is a unique $\beta_k \geq 0$ that satisfies $\rho(C_{\beta_k,F_k}) = 1$ by Proposition 3.4.1. Let $\beta_c := \max\{\beta_k : k = 1, 2, \ldots, n\}$. Then, $\beta_c \geq \beta_k$ implies that $\rho(C_{\beta_c,F_k}) \leq \rho(C_{\beta_k,F_k}) = 1$ and hence, $\rho(C_{\beta_c}) = 1$. Suppose that there exist a $\tilde{\beta}_c > 0$ with $\tilde{\beta}_c \neq \beta_c$ and $\rho(C_{\tilde{\beta}_c}) = 1$. Suppose without loss of generality that $\tilde{\beta}_c > \beta_c > 0$. For each $k = 1, 2, \ldots n$, we have that $1 \geq \rho(C_{\beta_c,F_k}) > \rho(C_{\tilde{\beta}_c,F_k})$ by the min-max version of the Collatz-Wielandt formula. This is a contradiction and therefore, β_c uniquely satisfies $\rho(C_{\beta_c}) = 1$. □

From this point on, if $\omega(e) > 0$ for all $e \in E^1$, then the critical inverse temperature is the unique β that satisfies $\rho(C_\beta) = 1$ and this is denoted by β_c.

Proposition 3.4.3. *Let E be a finite graph and $\beta \geq 0$ be such that $\rho(C_\beta) = 1$. Then, there exists a KMS$_\beta$ state of $(C^*(E), \alpha^\omega)$*

Proof. Let H be the set of sinks and decompose E^0 as $E^0\backslash H \cup H$. Then, we can write the matrix C_β as a block matrix

$$C_\beta = \begin{pmatrix} C_{\beta,E\backslash H} & F \\ 0 & 0 \end{pmatrix}, \tag{3.7}$$

and so, C_β is an upper triangular block matrix with $\rho(C_\beta) = \rho(C_{\beta,E\backslash H})$. Since $C_\beta \geq 0$, there exists a $z = (z_v)_{v\in E^0\backslash H}$ with $z \geq 0, z \neq 0$ and $||z||_1 = 1$ so that $C_{\beta,E\backslash H}z = z$ (see, for example, 8.3 of [42]). Let $x = (z \ \ 0)^{tr}$. Then, $||x||_1 = 1$ and $C_\beta x = x$. Hence, by Theorem 3.3.1, there exists a KMS$_\beta$ state that satisfies

$$\phi_x(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} x_{r(\mu)}. \quad \square$$

The following corollary shows the existence of a KMS state for quasi-free actions acting on finite graph algebras. This extends [29, Corollary 4.2] to quasi-free actions; in addition, the results show the existence of a KMS state for not only the Toeplitz algebra, but the graph algebra as well. Thus, there exists a KMS state of $(\mathcal{T}C^*(E), \alpha^\omega)$ that will always factor through a KMS state of $(C^*(E), \alpha^\omega)$. It is noteworthy that there are no restrictions on the structure of the graph E, such as strong connectivity.

Corollary 3.4.1. *Let E be a finite graph and $\omega(e) > 0$ for all $e \in E^1$. Then, there exists a KMS$_{\beta_c}$ state.*

Proof. A consequence of Proposition 3.4.2 and Proposition 3.4.3. $\square$

As a consequence of 3.4.1, we get the existence of a $\mathrm{KMS}_{\ln \rho(A)^{1/k}}$ state, when all the edges have label $k > 0$. In particular, when $k = 1$, the action reduces to the gauge action of the reals and we have the existence of a $\mathrm{KMS}_{\ln \rho(A)}$ state. This was exactly the critical inverse temperature described in [29].

Corollary 3.4.2. *Let E be a finite graph with at least one cycle and $\omega(e) = k > 0$ for all $e \in E^1$. Then, there exists a $KMS_{\ln \rho(A)^{1/k}}$ state.*

Proof. We note that $C_\beta = e^{-\beta k} A$ and $\beta_c = \ln \rho(A)^{1/k}$. Since E has at least one cycle, $\rho(A) \geq 1$ (see Appendix A of [29]). Thus, we have a $\mathrm{KMS}_{\ln \rho(A)^{1/k}}$ state by Corollary 3.4.1 above. □

3.5 KMS States above the Critical Inverse Temperature

In this section, we study the KMS states above a critical inverse temperature and extend the results of Theorem 3.1 in [29].

Theorem 3.5.1. *Let E be a finite directed graph and $C_\beta \in M_{E^0}(\mathbb{R})$ be the matrix defined by $C_\beta(v, w) = \sum_{e \in vE^1 w} e^{-\beta\omega(e)}$. Let α^ω be the quasi-free action corresponding to a labeling map ω, where $\omega(\mu) \neq 0$ for all $\mu \in E^* \backslash E^0$. Assume β is such that $\rho(C_\beta) < 1$.*

(a) For $v \in E^0$, the series $\sum_{\mu \in E^* v} e^{-\beta\omega(\mu)}$ either converges or is finite with sum $y_v \geq 1$. Set $y := (y_v) \in [1, \infty)^{E^0}$ and consider $\epsilon \in [0, \infty)^{E^0}$. Then, $m := (I - C_\beta)^{-1}\epsilon$ is a probability measure on E^0 if and only if $\epsilon \cdot y = 1$.

(b) Suppose $\epsilon \in [0, \infty)^{E^0}$ satisfies $\epsilon \cdot y = 1$ and $\epsilon_v = 0$ whenever v is a regular vertex. Then there is a KMS_β state ϕ_ϵ on $(C^*(E), \alpha^\omega)$ satisfying

$$\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} m_{r(\mu)}. \tag{3.8}$$

(c) The map $\epsilon \longmapsto \phi_\epsilon$ is an affine isomorphism of

$$\textstyle\sum_\beta := \{\epsilon \in [0,\infty)^{E^0} : \epsilon \cdot y = 1 \text{ and } \epsilon_v = 0 \text{ for } v \in E^0_{\mathrm{reg}}\}$$

onto the simplex of KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$. The inverse of this isomorphism takes the KMS$_\beta$ state ϕ to $(I - C_\beta)m^\phi$, where $m^\phi := (\phi(p_v))_v$.

Proof. (a) Let $v \in E^0$. Note that $C^n_\beta(w,v) = \sum_{\mu \in wE^n v} e^{-\beta\omega(\mu)}$. Then,

$$\sum_{\mu \in E^* v} e^{-\beta\omega(\mu)} = \sum_{n=0}^{\infty} \sum_{\mu \in E^n v} e^{-\beta\omega(\mu)} = \sum_{n=0}^{\infty} \sum_{w \in E^0} \sum_{\mu \in wE^n v} e^{-\beta\omega(\mu)} = \sum_{n=0}^{\infty} \sum_{w \in E^0} C^n_\beta(w,v). \tag{3.9}$$

Since $\rho(C_\beta) < 1$, the series $\sum_{n=0}^{\infty} C^n_\beta$ converges in the operator norm. Thus, for every fixed $w \in E^0$, the series $\sum_{n=0}^{\infty} C^n_\beta(w,v)$ converges and hence, the last sum in (3.9) converges. Also, since $C^0_\beta(v,v) = 1$, we have $y_v \geq 1$.

The expansion $m = \sum_{n=0}^{\infty} C^n_\beta \epsilon$ shows that $m \geq 0$ and

$$\begin{aligned} m(E^0) &= \sum_{v \in E^0} m_v = \sum_{v \in E^0} ((I - C_\beta)^{-1}\epsilon)_v \\ &= \sum_{v \in E^0} ((\sum_{n=0}^{\infty} C^n_\beta)\epsilon)_v = \sum_{v \in E^0} \sum_{n=0}^{\infty} \sum_{w \in E^0} C^n_\beta(v,w)\epsilon_w \\ &= \sum_{w \in E^0} \epsilon_w (\sum_{v \in E^0} \sum_{n=0}^{\infty} C^n_\beta(v,w)) = \sum_{w \in E^0} \epsilon_w (\sum_{\mu \in E^* w} e^{-\beta\omega(\mu)}) \\ &= \epsilon \cdot y. \end{aligned}$$

(b) By (a) we have a probability measure $m := (I - C_\beta)^{-1}\epsilon$ on E^0. Let v be a regular vertex. Then, $\epsilon_v = 0$ and we get that $m_v = (\sum_{n=0}^{\infty} C^{n+1}_\beta \epsilon)_v$ and

$$(C_\beta m)_v = (C_\beta (I - C_\beta)^{-1}\epsilon)_v$$

$$= \left(\left(\sum_{n=0}^{\infty} C_\beta^{n+1} \right) \epsilon \right)_v$$

$$= m_v.$$

Hence, by Theorem 3.3.1, there exists a KMS$_\beta$ state that satisfies (3.3).

(c) To see that every KMS$_\beta$ state ϕ has the form ϕ_ϵ, note that $m^\phi = (\phi(p_v))_{v \in E^0}$ satisfies Property P_β on E^0 and take $\epsilon := (I - C_\beta)m^\phi$. Then, $m := (I - C_\beta)^{-1}\epsilon = m^\phi$ shows that $\phi = \phi_\epsilon$. The formula (3.8) also shows that the map $F : \epsilon \longmapsto \phi_\epsilon$ is injective and that F is weak*-continuous from $\sum_\beta \subset \mathbb{R}^{E^0}$ to the state space of $C^*(E)$. To show that F is affine, let $\lambda \in (0, 1)$ and $\epsilon_1, \epsilon_2 \in \sum_\beta$ and let $\epsilon := \lambda\epsilon_1 + (1 - \lambda)\epsilon_2$. Let $m := (I - C_\beta)^{-1}\epsilon$, $m_1 := (I - C_\beta)^{-1}\epsilon_1$, and $m_2 := (I - C_\beta)^{-1}\epsilon_2$. Then, $m = \lambda m_1 + (1 - \lambda)m_2$ and $\phi_\epsilon = \lambda\phi_{\epsilon_1} + (1 - \lambda)\phi_{\epsilon_2}$. □

Remark 3.5.1. In part (b) of Theorem 3.5.1, we could have, instead, let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family that generates $C^*(E)$, $\pi : C^*(E) \to B(\mathcal{H})$ be a faithful nondegenerate representation and defined $P_v := \pi(p_v)$ and $S_e := \pi(s_e)$. For $\mu \in E^*$ we set $\Delta_\mu := e^{-\beta\omega(\mu)}\epsilon_{r(\mu)}$. Define

$$\phi_\epsilon(a) = \sum_{\mu \in E^*} \Delta_\mu(\pi(a)S_\mu | S_\mu) \text{ for } a \in C^*(E).$$

The rest follows from the argument in the proof of Theorem 3.1 (b) in [29].

Corollary 3.5.1. *Suppose β is such that $\rho(C_\beta) < 1$. Then, there exists a KMS$_\beta$ state if and only if E has a sink. If $|E^0_{\text{sinks}}| \neq 0$, then $\sum_\beta$ is a simplex of dimension $|E^0_{\text{sinks}}| - 1$.*

Proof. Suppose E is a graph with no sinks. Then, $\epsilon_v = 0$ for all $v \in E^0$. Thus, $\epsilon \cdot y \neq 1$ and by Theorem 3.5.1 (c) $\sum_\beta = \emptyset$ and there are no KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$. Suppose now that E has a sink v and define $\epsilon^v := (\epsilon^v_w) = (\delta_{w,v} y_v^{-1})$. Then, $\epsilon^v \cdot y = 1$, so there is a KMS$_\beta$ state ϕ_{ϵ^v}. We can also note that $\{\phi_{\epsilon^v}\}_{v \in E^0_{\text{sinks}}}$ are the set of all extreme points of the simplex of KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$. To see this note that the set $\sum_\beta$ is a polyhedral set in $\mathbb{R}^{E^0}$ with basic feasible solutions $\{\epsilon^v\}_{v\in E^0_{\text{sinks}}}$ (see Theorem 2.1.4 in Chapter 2 for the characterization of extreme points of a polyhedral set in terms of basic feasible solutions). □

Remark 3.5.2. The graph $E_{\mathcal{T}}$ is a graph with $|E^0|$ sinks. Hence, by Proposition 3.2.1, the simplex of KMS$_\beta$ states of $(\mathcal{T}C^*(E), \alpha^\omega)$ is of dimension $|E^0|-1$ (compare with remark 3.2 in [29]).

3.6 KMS States of a Graph with a Strongly Connected Subgraph

Below we give a complete description of the KMS states of $(C^*(E), \alpha^\omega)$, where E is a finite graph, H is the set of sinks in E^0 and $E\backslash H$ is strongly connected. As a consequence of Theorem 3.6.1, we extend the results of Theorem 4.3 in [29].

Theorem 3.6.1. *Let E be a finite graph with no sources and H be the set of sinks. Let $\omega(e) > 0$ for all $e \in E^1$. Suppose that $E\backslash H := (E^0\backslash H, E^1\backslash r^{-1}(H), r, s)$ is strongly connected and let $x = (y\ \ 0)^{tr}$, where $y = (y_v)_{v\in E^0\backslash H}$ is the unimodular Perron-Frobenius eigenvector of the matrix $C_{\beta_c, E\backslash H}$.*

(a) If $\beta > \beta_c$, then the set of all KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$ is a simplex of dimension $|H|-1$ and is affine-isomorphic to

$$\sum\nolimits_\beta := \{\epsilon \in [0,\infty)^{E^0} : \epsilon \cdot y = 1 \text{ and } \epsilon_v = 0 \text{ for } v \in E^0_{\text{reg}}\}.$$

(b) The system $(C^*(E), \alpha^\omega)$ has a unique KMS$_{\beta_c}$ state ϕ. This state satisfies

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} x_{r(\mu)} \tag{3.10}$$

and factors through a KMS_{β_c} *state* $\overline{\phi}$ *of* $(C^*(E\backslash H), \alpha^\omega)$.

(c) The state $\overline{\phi}$ *is the only KMS state of* $(C^*(E\backslash H), \alpha^\omega)$.

(d) If $\beta < \beta_c$*, then* $(C^*(E), \alpha^\omega)$ *has no* KMS_β *states.*

Proof. (a) Follows from Theorem 3.5.1 and Corollary 3.5.1.

(b) By Corollary 3.4.1, there exists a KMS_{β_c} state ϕ that satisfies (3.10). Suppose there exists another KMS_{β_c} state $\widetilde{\phi}$. Then, by Theorem 3.3.1, there exists an $\widetilde{x} \in \mathbb{R}^{E^0}$ that satisfies Property P_β on E^0, where C_{β_c} is of the form (3.7) with $\widetilde{x} = (\widetilde{y} \;\; z)^{tr}$ and $\widetilde{y} \in \mathbb{R}^{E^0\backslash H}$. Then, we have that $C_{\beta_c, E\backslash H}\widetilde{y} \leq C_{\beta_c, E\backslash H}\widetilde{y} + Fz = \widetilde{y}$. Since $\rho(C_{\beta_c}) = \rho(C_{\beta_c, E\backslash H})$, we have $C_{\beta_c, E\backslash H}\widetilde{y} = \widetilde{y}$ (see Theorem 1.6 of [53]). Thus, $Fz = 0$. Since E has no sources, we get that F has no zero columns and thus $z = 0$. Hence, $\widetilde{y}$ is the unimodular Perron-Frobenius eigenvector of the vertex matrix $C_{\beta_c, E\backslash H}$ and thus $\widetilde{\phi} = \phi$.

Suppose $H \neq \emptyset$. Since $E\backslash H$ is strongly connected, it contains a cycle and thus, $|E^0| \geq 2$. Let w be the basepoint of a cycle in E. Then, for every regular vertex $v \in E^0_{\text{reg}} = E^0\backslash H$, there is a path from v to w since $E\backslash H$ is strongly connected. This implies that $r(s^{-1}(v)) \subsetneq H$. Hence, $H = \overline{H}$ and so H is a saturated hereditary subset of E^0. Since $C^*(E)/I_H \cong C^*(E\backslash H)$ and $\phi(p_v) = 0$ for all $v \in H$, ϕ factors through a KMS_{β_c} state $\overline{\phi}$ of $(C^*(E\backslash H), \alpha^\omega)$ (see Lemma 2.2 of [28]).

(c) Follows immediately from Perron-Frobenius Theory and Theorem 3.3.1.

(d) Suppose ϕ is a KMS_β state of $(C^*(E), \alpha^\omega)$. Then, by Theorem 3.3.1, there exists a $y \in \mathbb{R}^{E^0\backslash H}$ such that $C_{\beta, E\backslash H}y \leq C_{\beta, E\backslash H}y + Fz = y$. Since $y \geq 0$, we have that $\rho(C_{\beta_c}) = 1 \geq \rho(C_\beta)$ by Theorem 1.6 of [53]. Hence, $\beta \geq \beta_c$. □

Remark 3.6.1. Suppose E is a strongly connected graph. If E consists of a single cycle, then there is a unique KMS_0 state by Proposition 3.4.1 and Theorem 3.6.1 above. Otherwise, $(C^*(E), \alpha^\omega)$ has no KMS_0 states.

Remark 3.6.2. If E is strongly connected, then $E_{\mathcal{T}}$ is a graph with no sources and $E_{\mathcal{T}} \backslash H = E$. Thus, Theorem 3.6.1 holds for $(\mathcal{T}C^*(E), \alpha^{\omega})$ when $\omega(e) > 0$ for all $e \in E^1$, by Proposition 3.2.1. In particular, Theorem 4.3 of [29] follows immediately as a consequence of Theorem 3.6.1, above.

3.7 KMS States on the Dual-Graph Algebra

In this section, we study the KMS states on the dual-graph algebra $C^*(\widehat{E})$ and find a correspondence to the KMS states on the graph algebra $C^*(E)$. Given a KMS state of one of the algebras, we are able to construct the corresponding KMS state of the other.

Definition 3.7.1. Let $E = (E^0, E^1, r, s)$ be a row-finite graph, H the set of sinks in E, and $\alpha^{\omega} : \mathbb{R} \curvearrowright C^*(E)$ be a quasi-free action, where ω is a labeling map on the edges E^1. Let $\beta \in \mathbb{R}$, $D_{\beta} = \text{diag}(e^{-\beta\omega(e)})_{e \in E^1}$ and $B \in M_{E^1}(\mathbb{N})$ the edge matrix of E. We say that a vector $y := (y_e)_{e \in E^1}$ satisfies Property P_{β} on E^1 if y is a probability measure on E^1 and $(D_{\beta}By)_e = y_e$ whenever $e \in E^1 \backslash r^{-1}(H)$.

Remark 3.7.1. We note that if a vector y satisfies Property P_{β} on E^1, then y satisfies the subinvariance relation $D_{\beta}By \leq y$.

Proposition 3.7.1. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph. Let $\omega : E^1 \to \mathbb{R}$ be a labeling map on E^1 that is bounded below. Define a labeling $\widehat{\omega} : E^2 \to \mathbb{R}$ on the edges of the dual graph by $\widehat{\omega}(ef) = \omega(e)$ for all $ef \in E^2$ and let $\eta^{\widehat{\omega}}$ be the corresponding quasi-free action on $C^*(\widehat{E})$. Suppose that $\widehat{\omega}(\widehat{\mu}) \neq 0$ for all $\widehat{\mu} \in \widehat{E}^* \backslash E^1$. For each $\beta \geq 0$, let $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ be the set of all* KMS$_{\beta}$ *states of $(C^*(\widehat{E}), \eta^{\widehat{\omega}})$ and let $L_{\beta,\eta^{\widehat{\omega}}} := \{y = (y_e)_{e \in E^1} : y$ satisfies Property P_{β} on $E^1\}$. Then, $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ is affine-isomorphic to $L_{\beta,\eta^{\widehat{\omega}}}$.*

Proof. The proof is similar to the argument in Theorem 3.3.1. Indeed, let $\{t_{ef}, q_e\}$ be the canonical Cuntz-Krieger $\widehat{E}$-famly that generates $C^*(\widehat{E})$. Let $\pi : C^*(\widehat{E}) \to B(\mathcal{H})$ be a faithful nondegenerate representation and define $Q_e := \pi(q_e)$ and $T_{ef} := \pi(t_{ef})$. Define a map $\Psi : \widehat{K}_{\beta,\eta} \to \mathbb{R}^{E^1}$, where $\Psi(\phi) = m^\phi$, where $m^\phi := (\phi(q_e))_{e\in E^1}$. Clearly, Ψ is an affine map that is weak*-continuous. Since ϕ is a state, we have that m^ϕ is a probability measure. Also, whenever e is a regular vertex, we have

$$\begin{aligned}
\phi(q_e) &= \sum_{ef\in E^2} \phi(t_{ef}t_{ef}^*) \\
&= \sum_{ef\in E^2} e^{-\beta\widehat{\omega}(ef)}\phi(t_{ef}^* t_{ef}) \\
&= \sum_{ef\in E^2} e^{-\beta\omega(e)}\phi(q_f) \\
&= \sum_{f\in E^1} e^{-\beta\omega(e)}B(e,f)\phi(q_f).
\end{aligned}$$

Thus, $(DBy^\phi)_e = y_e^\phi$ for $y^\phi = (\phi(q_e))_e$ and $y \in L_{\beta,\eta}$. If e is a sink , then $B(e,f) = 0$ for every vertex $f \in E^1$ and $(DBy^\phi)_e = \sum_{f\in E^1} DB(e,f)y_f^\phi = 0 \leq y_e^\phi$.

To show the image of Ψ is $L_{\beta,\eta}$, choose a $y \in L_{\beta,\eta}$. Define a tracial state τ on $C_0(E^1)$ by $\tau(a) = \sum_{e\in E^1} y_e(\pi|_{C_0(E^1)}(a)Q_e, Q_e)$. Indeed, $\tau(1) = \sum_{e\in E^1} y_e = 1$. If e is a a regular vertex, then

$$\begin{aligned}
\mathcal{F}_{\omega,\beta}(\tau)(q_e) &= \sum_{ef\in E^2} e^{-\beta\widehat{\omega}(ef)}\tau(q_f) \\
&= \sum_{ef\in E^2} e^{-\beta\omega(e)}y_f \\
&= \sum_{f\in E^1} e^{-\beta\omega(e)}B(e,f)y_f && (3.11) \\
&= y_e, && (3.12)
\end{aligned}$$

where (3.11) equals (3.12) since $y \in L_{\beta,\eta}$. By Theorem 3.2.1, we have a unique state

σ on the core of $C^*(\widehat{E})$ that satisfies (4.1) and $\sigma|_{C_0(E^1)} = \tau$ Hence, $\phi = \sigma \circ \Psi$ is a KMS_β state on $C^*(\widehat{E})$ by Theorem 3.3 of [14]. So,$\Psi(\phi)(e) = \phi(q_e) = \tau(q_e) = y_e$ and $\Psi(\phi) = y$.

To prove injectivity, suppose $\Psi(\phi_1) = \Psi(\phi_2)$. Then, $\phi_1(q_e) = \phi_2(q_e)$ for all $e \in E^1$. Hence, by Proposition 3.2 of [14], both KMS_β states coincide at its core. Since $\widehat{\omega}(\widehat{\mu}) \neq 0$ for all $\widehat{\mu} \in \widehat{E}^* \backslash E^1$, the KMS_β states are equal (3.4.1). $\square$

Remark 3.7.2. We could have instead applied Theorem 3.3.1 directly. Indeed, we have that $C_{\beta,\widehat{E},\widehat{\omega}}$ is a matrix in $M_{E^1}(\mathbb{R})$ and $C_{\beta,\widehat{E},\widehat{\omega}} = D_\beta B$, where $B \in M_{E^1}(\mathbb{N})$ is the edge matrix of E and $D_\beta = \text{diag}(e^{-\beta\omega(e)})_{e \in E^1}$.

Also, if we instead define a labeling map $\widehat{\omega} : E^2 \to \mathbb{R}$ on the edges of the dual graph by $\widehat{\omega}(ef) = \omega(f)$ for all $ef \in E^2$, we get $C_{\beta,\widehat{E},\widehat{\omega}} = BD_\beta$. We could apply Theorem 3.3.1 and obtain a similar affine-isomorphism as in Proposition 3.7.1. However, we will need the labeling map defined in Proposition 3.7.1 to find a correspondence between the KMS states on the graph algebra and dual-graph algebra.

Lemma 3.7.1. *Suppose E is a row-finite graph with no sinks and $\omega : E^1 \to \mathbb{R}$ a labeling of the edges of E. Define a labeling $\widehat{\omega} : E^2 \to \mathbb{R}$ on the edges of the dual graph by $\widehat{\omega}(ef) = \omega(e)$ for all $ef \in E^2$ and let $\eta^{\widehat{\omega}}$ be the corresponding quasi-free action on $C^*(\widehat{E})$. Then, $(C^*(\widehat{E}), \eta^{\widehat{\omega}}, \mathbb{R})$ is covariantly isomorphic to $(C^*(E), \alpha^\omega, \mathbb{R})$.*

Proof. Let $\{q_e, r_{ef}\}$ be the universal Cuntz-Krieger family for $\widehat{E}$ and $\{p_v, s_e\}$ the universal Cuntz-Krieger family for E. By Corollary 2.6 in [46], there is an isomorphism $\Phi : C^*(\widehat{E}) \to C^*(E)$ with $\Phi(q_e) = s_e s_e^*$ and $\Phi(r_{ef}) = s_e s_f s_f^*$. Then,

$$(\Phi \circ \eta_t)(q_e) = \Phi(q_e) = s_e s_e^* = (\alpha_t \circ \Phi)(q_e)$$

and

$$(\Phi \circ \eta_t)(r_{ef}) = e^{\widehat{\omega}(ef)it}\Phi(r_{ef}) = e^{\omega(e)it} s_e s_f s_f^* = (\alpha_t \circ \Phi)(r_{ef}). \quad \square$$

Theorem 3.7.1. *Suppose E is a row-finite graph without sinks and $\beta \geq 0$. Let $\omega : E^1 \to \mathbb{R}$ be bounded below and $\omega(\mu) \neq 0$ for all $\mu \in E^* \backslash E^0$. Then, $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ is affine-isomorphic to K_{β,α^ω}. Furthermore, if $x = (x_v)_{v \in E^0}$ is the corresponding vector for the KMS state ϕ_x in K_{β,α^ω}, then $y := (y_e)_{e \in E^1}$, where $y_e = \phi_x(s_e s_e^*)$ is the vector with corresponding KMS state $\widehat{\phi}_y$ in $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$. Conversely, if $y = (y_e)_{e \in E^1}$ is the vector with corresponding KMS state $\widehat{\phi}_y$ in $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$, then $x = (x_v)_{v \in E^0}$, where $x_v = \sum_{e \in vE^1} y_e$ is the corresponding vector for the KMS state ϕ_x in K_{β,α^ω}.*

Proof. By Lemma 3.7.1, the first part of the statement holds. For the second part, let $\widehat{\phi}_y$ be a KMS$_\beta$ state corresponding to $y \in L_{\beta,\eta^{\widehat{\omega}}}$. Let $\Phi : \widehat{K}_{\beta,\eta^{\widehat{\omega}}} \to K_{\beta,\alpha^\omega}; \widehat{\phi}_y \mapsto \phi_x$ be given by

$$\phi_x(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} x_{r(\mu)},$$

where $x_v = \sum_{e \in vE^1} y_e$. We have that $x_v \geq 0$ for every $v \in E^0$ and $\sum_{v \in E^0} x_v = 1$, since E is a row-finite graph with no sinks. For each $v \in E^0$,

$$\begin{aligned}
(C_\beta x)_v - x_v &= \sum_{w \in E^0} C_\beta(v,w) x_w - x_v = \sum_{w \in E^0} (\sum_{e \in vE^1 w} e^{-\beta\omega(e)} x_w) - x_v \\
&= \sum_{e \in vE^1} e^{-\beta\omega(e)} x_{r(e)} - x_v = \sum_{e \in vE^1} e^{-\beta\omega(e)} (\sum_{s(f)=r(e)} y_f) - x_v \\
&= \sum_{e \in vE^1} (\sum_{f \in E^1} D_\beta B(e,f) y_f) - x_v = \sum_{e \in vE^1} y_e - x_v \\
&= 0.
\end{aligned}$$

So, by Theorem 3.3.1, we have that ϕ_x is a KMS$_\beta$ state and thus, Φ is a well-defined map.

To show Φ is injective, suppose $\phi_x = \phi_{\widetilde{x}}$. Then, $\phi_x(s_e s_e^*) = e^{-\beta\omega(e)} x_{r(e)} = y_e$ and similarly, $\phi_{\widetilde{x}}(s_e s_e^*) = \widetilde{y}_e$. Since $\omega(\mu) \neq 0$ for all $\mu \in E^* \backslash E^0$, we have that $\widehat{\omega}(\widehat{\mu}) \neq 0$ for all $\widehat{\mu} \in \widehat{E}^* \backslash E^1$. By Proposition 3.7.1, $\widehat{\phi}_y = \widehat{\phi}_{\widetilde{y}}$, since $y_e = \widetilde{y}_e$ for all $e \in E^1$ and y

is a probability measure on E^1.

To prove surjectivity, suppose ϕ is a KMS$_\beta$ of $(C^*(E), \alpha^\omega)$. Let $y_e := \phi(s_e s_e^*)$ and $y := (y_e)_{e\in E^1}$. Then, y satisfies Property P_β on E^1. Hence, by Proposition 3.7.1, there exists KMS$_\beta$ state $\widehat{\phi}_y$ of $(C^*(\widehat{E}), \eta^{\widehat{\omega}})$. Since $\phi(p_v) = \sum_{s(e)=v} \phi(s_e s_e^*) = \sum_{s(e)=v} y_e$, we have that $\Phi(\widehat{\phi}_y) = \phi_x = \phi$. We have that Φ is an affine and weak*-continuous map from $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ to K_{β,α^ω}. Therefore, Φ is the affine-isomorphism and hence, the correspondence follows as desired. $\square$

3.8 Ground States and KMS$_\infty$ States

Proposition 3.8.1. *Let E be a finite directed graph and let $\alpha^\omega : \mathbb{R} \to \mathrm{Aut}C^*(E)$ be the quasi-free action corresponding to a labeling map ω, where $\omega(e) > 0$ for all $e \in E^1$. Suppose that ϵ is a probability measure on E^0. Then, there is a KMS$_\infty$ state ϕ_ϵ satisfying*

$$\phi_\epsilon(s_\mu s_\nu^*) = \begin{cases} 0 & \textit{unless } |\mu| = |\nu| = 0 \textit{ and } \mu = \nu \\ \epsilon_v & \textit{if } \mu = \nu = v \in E^0. \end{cases}$$

Every ground state of $(C^*(E), \alpha^\omega)$ is a KMS$_\infty$ state and the map $\epsilon \longmapsto \phi_\epsilon$ is an affine isomorphism of the simplex of probability measures of E^0 onto the set of ground states of $(C^*(E), \alpha^\omega)$.

Proof. Choose a sequence $\beta_j \to \infty$ as $j \to \infty$ with $\beta_j > \beta_c$. For each j, define (y_v^j) as in Theorem 3.5.1 (a) by $y_v^j = \sum_{\mu\in E^*v} e^{-\beta_j \omega(\mu)}$. Set $\epsilon_v^j := \epsilon_v (y_v^j)^{-1}$, and let ϕ_j be the KMS$_{\beta_j}$ state ϕ_{ϵ^j} of $(C^*(E), \alpha^\omega)$, described in Theorem 3.5.1(b). We note that if we choose an $M > 0$ so that $\beta_j > M > \beta_c$ for all j sufficiently large, we can apply the Dominated Convergence Theorem and prove that $y_v^j \to 1$ as $j \to \infty$. The rest follows from the argument in Theorem 5.1 of [29]. $\square$

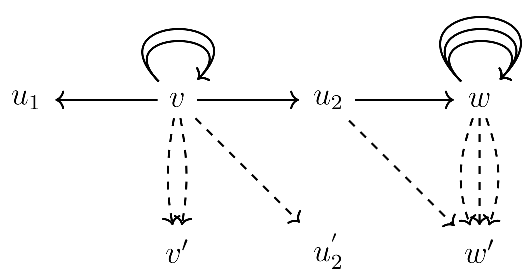
3.9 Examples

In the examples below, we calculate the KMS states of $(C^*(E), \alpha^\omega)$ and $(\mathcal{T}C^*(E), \alpha^\omega)$. By Proposition 3.2.1, finding the KMS states of $(\mathcal{T}C^*(E), \alpha^\omega)$ reduces to finding the KMS states of $(C^*(E_{\mathcal{T}}), \alpha^\omega)$.

We then can compute the KMS states by finding all elements in the polyhedral set L_{β,α^ω} (see (3.6) in Remark 3.3.2). Calculating the extreme points in L_{β,α^ω} coincides with finding all basic feasible solutions (see Theorem 2.6.4 of [3]).

Below, the solid lines represent all of the edges in the graph E and the solid lines along with the dashed lines represent all of the edges of the graph $E_{\mathcal{T}}$.

Example 3.9.1. In this example, we calculate the KMS state for the gauge action γ of $\mathbb{R}$, where the graph E has one sink and $E_{\mathcal{T}}$ has many more sinks. This example was computed in Example 6.4 of [28] using strongly connected components, but we calculate it here using Theorem 3.3.1.



KMS$_\beta$ states of $(C^*(E), \gamma)$:

$$\tilde{R}_\beta = \begin{pmatrix} e^\beta - 2 & -1 & 0 & -1 \\ 0 & e^\beta & -1 & 0 \\ 0 & 0 & e^\beta - 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- For $\beta > \ln 3$, we have a unique KMS$_\beta$ state that corresponds to the vector $m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta - 1} & 0 & 0 & \frac{e^\beta - 2}{e^\beta - 1} \end{pmatrix}^{tr}$.

- For $\beta = \ln 3$, we have a 2-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 \end{pmatrix}^{tr}$ and $m_2^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}^{tr}$.

- For $\ln 2 < \beta < \ln 3$, we have a unique KMS$_\beta$ state that corresponds to the vector $m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta - 1} & 0 & 0 & \frac{e^\beta - 2}{e^\beta - 1} \end{pmatrix}^{tr}$.

- For $\beta = \ln 2$, we have a unique KMS$_\beta$ state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{tr}$.

- For $\beta < \ln 2$, no solution exists and therefore, there are no KMS$_\beta$ states.

KMS$_\beta$ states of $(C^*(E_\mathcal{T}), \gamma)$:

$$\tilde{R}_\beta = \begin{pmatrix} e^\beta - 2 & -1 & 0 & -1 & -2 & -1 & 0 \\ 0 & e^\beta & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & e^\beta - 3 & 0 & 0 & 0 & -3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

- For $\beta > \ln 3$, we have a 3-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta - 1} & 0 & 0 & \frac{e^\beta - 2}{e^\beta - 1} & 0 & 0 & 0 \end{pmatrix}^{tr}$, $m_2^\beta = \begin{pmatrix} \frac{2}{e^\beta} & 0 & 0 & \frac{e^\beta - 2}{e^\beta} & 0 & 0 & 0 \end{pmatrix}^{tr}$, $m_3^\beta = \begin{pmatrix} \frac{1}{e^\beta - 1} & 0 & 0 & 0 & 0 & \frac{e^\beta - 2}{e^\beta} & 0 \end{pmatrix}^{tr}$, and $m_4^\beta = \begin{pmatrix} \frac{1}{e^{2\beta} - e^\beta - 1} & \frac{e^\beta - 2}{e^{2\beta} - e^\beta - 1} & \frac{3e^\beta - 6}{e^{2\beta} - e^\beta - 1} & 0 & 0 & 0 & 0 & \frac{e^{2\beta} - 5e^\beta + 6}{e^{2\beta} - e^\beta - 1} \end{pmatrix}^{tr}$.

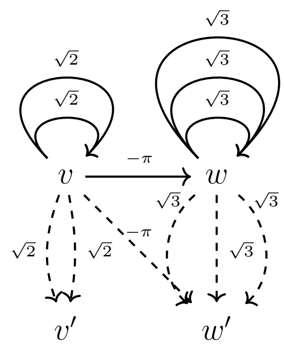
- For $\beta = \ln 3$, we have 3-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 \cdots 0 \end{pmatrix}^{tr}$, $m_2^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}^{tr}$, $m_3^\beta = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}^{tr}$, and $m_4^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}^{tr}$.

- For $\ln 2 < \beta < \ln 3$, we have a unique KMS$_\beta$ state that corresponds to the vector $m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta - 1} & 0 & 0 & \frac{e^\beta - 2}{e^\beta - 1} & 0 & 0 & 0 \end{pmatrix}^{tr}$.

- For $\beta = \ln 2$, we have a unique KMS$_\beta$ state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 \cdots 0 \end{pmatrix}^{tr}$.

- For $\beta < \ln 2$, no solution exists and therefore there are no KMS$_\beta$ states.

Example 3.9.2. In this example, we find the KMS states for a quasi-free action that is not the gauge action with labels that are both negative and positive.



KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$:

$$R_\beta = \begin{pmatrix} 1 - 2e^{-\beta\sqrt{2}} & -e^{\beta\pi} \\ 0 & 1 - 3e^{-\beta\sqrt{3}} \\ 1 & 1 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- For $\beta > \frac{\ln 3}{\sqrt{3}}$, no solution exists and therefore, there are no KMS$_\beta$ states.

- For $\beta = \frac{\ln 3}{\sqrt{3}}$, we have a unique KMS$_\beta$ state corresponding to the vector $m_1^\beta = \begin{pmatrix} \frac{1}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & \frac{\xi^\pi(1-2\xi^{\sqrt{2}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} \end{pmatrix}^{tr}$, where $\xi = (\frac{1}{3})^{\frac{1}{\sqrt{3}}}$.

- For $\frac{\ln 2}{\sqrt{2}} < \beta < \frac{\ln 3}{\sqrt{3}}$, no solution exists and therefore, there are no KMS$_\beta$ states.

- For $\beta = \frac{\ln 2}{\sqrt{2}}$, we have a unique KMS$_\beta$ state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 \end{pmatrix}^{tr}$.

- For $\beta < \frac{\ln 2}{\sqrt{2}}$, no solution exists and therefore, there are no KMS$_\beta$ states.

KMS_β states of $(C^*(E_\mathcal{T}), \alpha^\omega)$:

$$R_\beta = \begin{pmatrix} 1-2e^{-\beta\sqrt{2}} & -e^{\beta\pi} & -2e^{-\beta\sqrt{2}} & -e^{\beta\pi} \\ 0 & 1-3e^{-\beta\sqrt{3}} & 0 & -3e^{-\beta\sqrt{3}} \\ 1 & 1 & 1 & 1 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- For $\beta > \frac{\ln 3}{\sqrt{3}}$, we have a 1-dimensional simplex of KMS_β states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} 2\xi^{\sqrt{2}} & 0 & 1-2\xi^{\sqrt{2}} & 0 \end{pmatrix}^{tr}$, and $m_2^\beta = \begin{pmatrix} \frac{1}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & \frac{3\xi^{\sqrt{3}+\pi}(1-2\xi^{\sqrt{2}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & 0 & \frac{\xi^\pi(1-2\xi^{\sqrt{2}})(1-3\xi^{\sqrt{3}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} \end{pmatrix}^{tr}$,where $\xi = e^{-\beta}$.

- For $\beta = \frac{\ln 3}{\sqrt{3}}$, we have a 1-dimensional simplex of KMS_β states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & \frac{\xi^\pi(1-2\xi^{\sqrt{2}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & 0 & 0 \end{pmatrix}^{tr}$ and $m_2^\beta = \begin{pmatrix} 2\xi^{\sqrt{2}} & 0 & 1-2\xi^{\sqrt{2}} & 0 \end{pmatrix}^{tr}$, where $\xi = (\frac{1}{3})^{\frac{1}{\sqrt{3}}}$.

- For $\frac{\ln 2}{\sqrt{2}} < \beta < \frac{\ln 3}{\sqrt{3}}$, we have a unique KMS_β state that corresponds to the vector $m_1^\beta = \begin{pmatrix} 2\xi^{\sqrt{2}} & 0 & 1-2\xi^{\sqrt{2}} & 0 \end{pmatrix}^{tr}$, where $\xi = e^{-\beta}$.

- For $\beta = \frac{\ln 2}{\sqrt{2}}$, we have a unique KMS_β state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{tr}$.

- For $\beta < \frac{\ln 2}{\sqrt{2}}$, no solution exists and therefore, there are no KMS_β states.

Chapter 4

The Structure of Crossed Products of Graph Algebras

In this chapter, the structure of crossed products of graph algebras by quasi-free actions is investigated. The main results show that the crossed product $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ can be written as an inductive limit of non-unital one-dimensional NCCW complexes for at least some dense G_δ set in a subspace of $\mathbb{R}^{E^1}$.This extends the results in [15], where the Cuntz algebra case was studied. In this chapter, the result is first proven for the finite graph case (Theorem 4.4.1) and then, as a consequence, the result holds for row-finite (possibly infinite) graphs (Theorem 4.4.2). As in [15], the basis of our construction is viewing these crossed products as fibres in a continuous field over $\mathbb{R}^{E^1}$. The K-theory of certain AF algebras associated to the rational fibres is computed and these results are applied to the Cuntz algebra case (see Proposition 4.5.1 and Theorem 4.5.1).

4.1 The Cuntz Algebra Case

In [15, Theorem 5.1], it was shown that for at least a dense G_δ set of labels, the crossed product of a Cuntz algebra by a quasi-free action can be written as an inductive limit of non-unital one-dimensional NCCW complexes. The special case of O_2 was considered in order to simplify calculations and book-keeping, however, the general argument also extends to O_n.

Let $\{s_1, s_2\}$ be the canonical generating set for O_2. The quasi-free actions on the Cuntz-algebra O_2 are denoted α^λ, where $\alpha_t^\lambda(s_1) = e^{2\pi i\lambda_1 t}s_1$ and $\alpha_t^\lambda(s_2) = e^{2\pi i\lambda_1 t}s_2$, where λ_1 and λ_2 are real numbers. In [15], the real parameter was scaled to assume $\lambda_1 = 1$. This allowed for a one-parameter family of crossed products $O_2 \rtimes_{\alpha^\lambda} \mathbb{R}$, where $\lambda = \lambda_2$. In the case of O_n, there would be an $n-1$ parameter family of crossed products $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$, where $\lambda = (\lambda_2, \ldots, \lambda_n)$. The case having all positive labels was considered. Briefly, the general method used in [15] is as follows:

- The crossed products $O_2 \rtimes_{\alpha^\lambda} \mathbb{R}$ were viewed as fibres in a continuous field of C^*-algebras.

- The structure of the rational fibres, namely $O_2 \rtimes_{\alpha^\lambda} \mathbb{R}$, where $\lambda = \frac{p}{q} > 0$, was investigated. Using a change of variables, this reduces to studying the new action α that satisfies $\alpha_t(s_1) = e^{2\pi iqt}s_1$and $\alpha_t(s_2) = e^{2\pi ipt}s_2$. The crossed product of $O_2 \rtimes_\alpha \mathbb{R}$ was shown to be isomorphic to the mapping torus of $O_2 \rtimes_\alpha \mathbb{T}$ by an automorphism generating the dual action of $\mathbb{Z}$ [15, Theorem 3.1]. In particular, Dean showed that $O_2 \rtimes_\alpha \mathbb{R}$ is isomorphic to the mapping torus of a simple AF algebra $A(p,q) \cong O_2 \rtimes_\alpha \mathbb{T}$ by $\hat{\alpha}$. Here, $A(p,q)$ was a universal C^*-algebra given by generators and relations.

- The mapping torus discussed above was written as an inductive limit of non-unital one-dimensional NCCW complexes [15, Corollary 3.5]. That is, all positive rational fibres were written as inductive limits of non-unital one-dimensional NCCW complexes.

- The rational fibres satisfy a local approximation property. By stable relations and a Baire category argument, it follows that a dense G_δ set of the fibres have this local approximation property. Since they satisfy a local approximation property, they can be written as inductive limits of non-unital one-dimensional NCCW complexes ([15, Lemma 4.6]).

4.2 The Graph Algebra Case

The main result of this chapter extends the results of [15, Theorem 5.1] to row-finite graph algebras. The case for finite graph algebras is proved in Theorem 4.4.1 and then, as a consequence, the row-finite case is addressed in Theorem 4.4.2. The general method is described below:

- Let E be a graph and $c : E^1 \to \mathbb{R}$ a labeling map giving a periodic action α^c on $C^*(E)$. If E is a finite graph, we can assume that all the labels are integers, by scaling the parameters as described above. If E is an infinite graph, then the labels will be integers, except for possibly finitely many distinct rational numbers. In this case, we can also assume that all the labels are integers, by scaling the parameters as well. From now on, when α^c is referred to as a periodic action, the labeling map is $c : E^1 \to \mathbb{Z}$.

- The 'rational' fibres are viewed as mapping tori over skew-product graph algebras (see Section 4.3). In Proposition 4.3.3, the 'rational' (periodic) fibres $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ are shown to be inductive limits of non-unital 1-NCCW complexes, extending [15, Corollary 3.5] to arbitrary graph algebras.

- The rational fibres satisfy a local approximation property. By stable relations and a Baire category argument, it follows that a dense G_δ set of the fibres have this local approximation property. Since they satisfy a local approximation property, they can be written as inductive limits of non-unital one-dimensional NCCW complexes.The finite graph algebra case is proved in Theorem 4.4.1 and further extended to row-finite graph algebras in Theorem 4.4.2.

- In Section 4.5, the K-Theory of the AF algebras used in the construction of the mapping torus is calculated for the case when the graph has no sinks and the labeling map gives all positive integers or all negative integers (Proposition 4.5.1). Also, the ordered K_0-group is calculated for the Cuntz algebra case (Theorem 4.5.1).

4.3 Skew-Product Graph

General theory for skew-product graphs can be found in [37] and [31]. The skew-product graph defined below is the same as the one described in [31]. The skew-product graphs used in [37], although defined differently, are isomorphic to the ones used in this thesis ([31, Remark 2.2]).

Let E be a graph and G be a countable group. Given a labeling map $c : E^1 \to G$, we define the skew-product graph, denoted $E \times_c G$, to be the graph having vertex set $E^0 \times G$, edge set $E^1 \times G$, and with range and source maps defined by $r(e, g) = (r(e), g)$ and $s(e, g) = (s(e), c(e)g)$ for $(e, g) \in E^1 \times G$. Note that $E \times_c G$ is row-finite if and only if E is row-finite. Also, (v, g) is a sink if and only if v is a sink.

The skew-product graph is an AF algebra if and only if $c(\mu) \neq 1_G$ for every cycle $\mu \in E$. Indeed, (μ, g) is a cycle in $E \times_c G$ if $r(\mu, g) = s(\mu, g)$, which implies that $(r(\mu), g) = (s(\mu), c(\mu)g)$. Hence, (μ, g) is a cycle in $E \times_c G$ if and only if μ is a cycle in E and $c(\mu) = 1_G$.

The group G acts on the skew-product graph via right translation:

$$g \cdot (v, h) = (v, hg^{-1})$$
$$g \cdot (e, h) = (e, hg^{-1}).$$

This induces an action $\beta : G \curvearrowright C^*(E \times_c G)$ such that

$$\beta_g(s_{(e,h)}) = s_{(e,hg^{-1})}$$
$$\beta_g(p_{(v,h)}) = p_{(v,hg^{-1})}$$

(see [31]). Below, we will use $G = \mathbb{Z}$. Then, we have the skew-product graph $E \times_c \mathbb{Z}$, with range and source maps as follows:

$$s(e,n) = (s(e), n - c(e)) \text{ and } r(e,n) = (r(e), n).$$

The induced action $\beta : \mathbb{Z} \to \mathrm{Aut} C^*(E \times_c \mathbb{Z})$ satisfies $\beta_m(s_{(e,n)}) = s_{(e,n+m)}$ and $\beta_m(p_{(v,n)}) = p_{(v,n+m)}$.

The proposition below will be useful in analyzing the crossed products of graph algebras by periodic quasi-free actions.

Proposition 4.3.1. ([46, Lemma 7.10],[31, Theorem 2.4]) *Let E be a row-finite directed graph. Then, there is an isomorphism Φ of $C^*(E \times_c \mathbb{Z})$ onto $C^*(E) \rtimes_\alpha \mathbb{T}$ such that $\Phi \circ \beta_m = \widehat{\alpha}_m \circ \Phi$.*

Proposition 4.3.2 below will be useful for the proof of Proposition 4.3.3, where the skew-product graph algebra is written as an inductive limit of finite-dimensional C^*-algebras.

Definition 4.3.1. [43, Definition 3.6] Let $E = (E^0, E^1, r, s)$ be a graph and let $F = (F^0, F^1, r_F, s_F)$ be a subgraph of E. Define a graph $E_F = (E_F^0, E_F^1, r_{E_F}, s_{E_F})$ as follows. Set $S := \{v \in F^0 : |s_F^{-1}(v)| < \infty, \emptyset \subsetneq s_F^{-1}(v) \subsetneq s_E^{-1}(v)\}$, and let

$$E_F^0 := F^0 \cup \{v' : v \in S\} \text{ and } E_F^1 := F^1 \cup \{e' : e \in F^1 \text{ and } r(e) \in S\},$$

with range and source maps given by

$$s_{E_F}(e) = s(e),\ s_{E_F}(e') = s(e),\ r_{E_F}(e) = r(e),\ r_{E_F}(e') = r(e)'.$$

Proposition 4.3.2 follows from [43, Theorem 3.7] as discussed in [43, Example 3.8] and shows that the C^*-subalgebra of $C^*(E)$ generated by elements that come from a subgraph F is isomorphic to a graph algebra whose corresponding graph E_F is defined above.

Proposition 4.3.2. ([43, Theorem 3.7, Example 3.8]) *Let E be a graph, $\{s_e, p_v\}$ be a generating Cuntz-Krieger E-family in $C^*(E)$, and F be a subgraph of E. Then, the C^*-subalgebra of $C^*(E)$ generated by $\{s_e : e \in F^1\} \cup \{p_v : v \in F^0\}$, denoted by $C^*(\{s_e : e \in F^1\} \cup \{p_v : v \in F^0\})$, is isomorphic to $C^*(E_F)$. Furthermore, if we define*

$$q_w := \begin{cases} p_v & \text{if } w \in F^0 \backslash S \\ \sum_{\{e \in F^1 : s(e) = w\}} s_e s_e^* & \text{if } w \in S \\ p_v - \sum_{\{e \in F^1 : s(e) = v\}} s_e s_e^* & \text{if } w = v' \text{ for some } v \in S \end{cases} \tag{4.1}$$

$$t_f := \begin{cases} s_f q_{r(f)} & \text{if } f \in F^1 \\ s_e q_{r(e)'} & \text{if } f = e' \text{ for some } e \in F^1, \end{cases} \tag{4.2}$$

then $\{t_f, q_w\}$ will be a generating Cuntz-Krieger E_F-family in $C^*(\{s_e : e \in F^1\} \cup \{p_v : v \in F^0\})$.

The main result of this chapter deals with writing crossed products as inductive limits of NCCW-complexes. The definition is provided below.

Definition 4.3.2. [45, Definition 11.2] A zero dimensional NCCW-complex is any finite dimensional C^*-algebra A_0. An n-dimensional NCCW-complex is defined as any C^*-algebra A_n, which is a pull-back of the following diagram:

$$\begin{array}{ccc} & & A_{n-1} \\ & & \big\downarrow \phi_n \\ C([0,1]^n, F_n) & \xrightarrow{\delta} & C(S^{n-1}, F_n) \end{array}$$

where A_{n-1} is an $(n-1)$-dimensional-NCCW complex, F_n is some finite dimensional C^* algebras, δ is the boundary restriction map, and ϕ_n is an arbitrary morphism called the connecting morphism. An NCCW complex A_n is called unital if if A_{n-1} is unital and the connecting morphism ϕ_n is also unital. A NCCW complex is called non-unital if it is not necessarily unital.

In this thesis, we construct a one-dimensional NCCW complex in the following way. Let F_0 and F_1 be two finite dimensional C^*-algebras with maps $\alpha_1, \alpha_2 : F_0 \to F_1$. Let ev(0), ev(1) denote the maps from $F_1 \otimes C[0,1]$ to F_1, given by evaluation at zero and one, respectively. Then, we get a *non-unital one-dimensional NCCW-complex* as the pull-back of the following diagram:

$$\begin{array}{ccc} & & F_0 \\ & & \big\downarrow \alpha_1 \oplus \alpha_2 \\ F_1 \otimes C[0,1] & \xrightarrow{\mathrm{ev}(0) \oplus \mathrm{ev}(1)} & F_1 \oplus F_1 \end{array}$$

The first part of the construction is to analyze the rational fibres and write these crossed products as inductive limits of non-unital one-dimensional NCCW-complexes.

Proposition 4.3.3. *Let E be a finite graph and let α^c be a periodic action on $C^*(E)$ with corresponding labeling map c. If $c(\mu) \neq 0$ for any cycle $\mu \in E^*$, then $C^*(E) \rtimes_{\alpha^c} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes.*

Proof. Since α^c is a periodic action, by rescaling we may assume $c : E^1 \to \mathbb{Z}$. Let

$E \times_c \mathbb{Z} := (E^0 \times_c \mathbb{Z}, E^1 \times_c \mathbb{Z}, r, s)$ be the skew-product graph. By Proposition 4.3.1, we have that there is an isomorphism Φ of $C^*(E \times_c \mathbb{Z})$ onto $C^*(E) \rtimes_\alpha \mathbb{T}$ with $\Phi \circ \beta_m = \widehat{\alpha}_m \circ \Phi$.

Let $E = (E^0, E^1, r, s)$ be a finite graph and define $K := \{v \in E^0 : r^{-1}(v) = \emptyset, s^{-1}(v) = \emptyset\}$. For each n, define a subgraph F_n of $E \times_c \mathbb{Z}$ by $F_n^0 = \{r(e,k) : e \in E^1, -n \leq k \leq n\} \cup \{s(e,k) : e \in E^1, -n \leq k \leq n\} \cup \{(v,k) : v \in K, -n \leq k \leq n\}$ and $F_n^1 = \{(e,k) : e \in E^1, -n \leq k \leq n\}$. Then, $F_n \subset F_{n+1}$ and $E \times_c \mathbb{Z} = \bigcup_n F_n$.

Let $\{s_{(e,k)}, p_{(v,k)} : e \in E^1, v \in E^0, k \in \mathbb{Z}\}$ be the canonical Cuntz-Krieger family generating $C^*(E \times_c \mathbb{Z})$. Define A_n to be C^*-subalgebra of $C^*(E \times_c \mathbb{Z})$ generated by $\{s_{(e,k)} : (e,k) \in F_n^1\} \cup \{p_{(v,k)} : (v,k) \in F_n^0\}$. Then, $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of C^*-subalgebras of $C^*(E \times_c \mathbb{Z})$ with

$$C^*(E \times_c \mathbb{Z}) = \overline{\bigcup_{n \geq 1} A_n}.$$

Since $c(\mu) \neq 0$ for any cycle $\mu \in E^*$, $E \times_c \mathbb{Z}$ has no cycles. Thus, each A_n is isomorphic to a finite graph algebra (see Proposition 4.3.2), in which the graph has no cycles. So, A_n is finite dimensional.

Lastly, since A_n and $\beta(A_n)$ are both included into A_{n+1}, we can now define the non-unital NCCW-complex B_n, as in [15]. That is,

$$B_n = \{f \in C([0,1], A_{n+1}) : f(0) \in A_n, \beta(f(0)) = f(1)\}.$$

Then, $B_n \subseteq B_{n+1}$ for all n and

$$\begin{aligned} C^*(E) \rtimes_\alpha \mathbb{R} &\cong M_{\widehat{\alpha}}(C^*(E) \rtimes_\alpha \mathbb{T}) \\ &\cong M_\beta(C^*(E \times_c \mathbb{Z})) \\ &= \overline{\bigcup_n B_n}. \end{aligned}$$

4.4 The Structure of the Crossed Product

Definition 4.4.1. We say that a C^*-algebra A has the local approximation property with respect to the class of C^*-algebras $\mathfrak{C}$ if, for every finite set F of elements of A and every $\epsilon > 0$, there is a $C \in \mathfrak{C}$ and a $*$-homomorphism $\phi : C \to A$ such that each element of F lies within ϵ of the image of ϕ.

Lemma 4.4.1. [52, Proposition 6 (xiii)]*If a separable C^*-algebra has the local approximation property with respect to the class of non-unital one-dimensional NCCW-complexes, then it is an inductive limit of a sequence of non-unital one-dimensional NCCW-complexes.*

Suppose $E = (E^0, E^1, r, s)$ is a finite graph with edges $E^1 = \{e_1, e_2, \ldots, e_m\}$. From now on, we will denote $L^m := \{(\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_m}) \in \mathbb{Q}^m : \lambda$ is a labeling map on E^* and $\lambda_\mu \neq 0$ for any cycle $\mu \in E^*\}$.

Corollary 4.4.1. *Let E be a finite graph with edges $E^1 = \{e_1, e_2, \ldots, e_m\}$ and $\lambda_0 := (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_m}) \in L^m$ so that α^{λ_0} is a periodic action of $\mathbb{R}$ on $C^*(E)$. Suppose further that $\epsilon > 0$ and $f_1, \ldots, f_n \in \mathrm{C}_c(\mathbb{R}, C^*(E))$. Then, there exists a neighbourhood U of λ_0 in $\mathbb{R}^m$, a non-unital one-dimensional NCCW-complex A, and for every $s \in U$, a $*$-homomorphism $\psi : A \to C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ such that $\{\varphi_s(f_1), \ldots, \varphi_s(f_n)\} \subseteq_\epsilon \psi_s(A)$, where φ_s denotes the canonical inclusion of $\mathrm{C}_c(\mathbb{R}, C^*(E))$ into $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$.*

Proof. Suppose $\lambda_0 \in \mathbb{R}^m$, $\epsilon > 0$ and $f_1, \ldots, f_n \in \mathrm{C}_c(\mathbb{R}, C^*(E))$. Let φ_{λ_0} denotes the canonical inclusion of $\mathrm{C}_c(\mathbb{R}, C^*(E))$ into $C^*(E) \rtimes_{\alpha^{\lambda_0}} \mathbb{R}$. By Proposition 4.3.3, $C^*(E) \rtimes_{\lambda_0} \mathbb{R} \cong \overline{\bigcup_n B_n}$, where B_n are non-unital one-dimensional NCCW complexes. Thus, by choosing n large enough, there exists a $*$-homomorphism $\psi : B_n \to C^*(E) \rtimes_{\lambda_0} \mathbb{R}$ with $\{\varphi_{\lambda_0}(f_1), \varphi_{\lambda_0}(f_2), \ldots, \varphi_{\lambda_0}(f_n)\} \subseteq_{\epsilon/2} C^*(E) \rtimes_{\lambda_0} \mathbb{R}$. The rest follows from [15, Lemma 4.8]. □

Theorem 4.4.1. (Finite Graph Case) *Let E be a finite graph with edges $E^1 = \{e_1, e_2, \dots, e_m\}$. Then, the set of points $\lambda \in \overline{L^m}$, for which $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes, contains a dense G_δ set in $\overline{L^m}$. For such λ, the crossed product $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable.*

Proof. The first statement follows from the same argument as in Theorem 4.10 of [15]. As in the proof of [15, Theorem 4.10], we need to show that the local approximation property with respect to the class of non-unital one-dimensional NCCW-complexes holds for such a set. Pick a countable dense subset of $C_c(\mathbb{R}, C^*(E))$ and call it $\mathfrak{D}$. Let φ_s be the canonical inclusion of $C_c(\mathbb{R}, C^*(E))$ into $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$. From Corollary 4.4.1 above, for each finite subset $F \subset \mathfrak{D}$, $\epsilon > 0$ and $\lambda \in L^m$, there is a neighbourhood $V(\lambda, F, \epsilon)$ of λ, a non-unital one-dimensional NCCW-complex $B(\lambda, F, \epsilon)$, and for every $s \in V(\lambda, F, \epsilon)$ a $*$-homomorphism $\psi(\lambda, s, F, \epsilon) : B(\lambda, F, \epsilon) \to C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ such that $\varphi_s(F) \subseteq_\epsilon \psi(\lambda, s, F, \epsilon) B(\lambda, F, \epsilon)$. Let $G(\epsilon, F) = \bigcup_{\lambda \in L^m} V(\lambda, F, \epsilon)$. Then, for every s in $G(\epsilon, F)$, $\varphi_s(F)$ is approximately contained to within ϵ by the image of a non-unital one-dimensional NCCW complex. Also, $G(\epsilon, F)$ contains a dense open set in $\overline{L^m}$. Let ϵ_n be a sequence of positive numbers converging to zero and let $\mathfrak{F}(\mathfrak{D})$ denote the set of finite subsets of $\mathfrak{D}$. Then the set $G = \bigcap_{F \in \mathfrak{F}(\mathfrak{C})} \bigcap_{\epsilon_n} G(\epsilon_n, F)$ is contained in the set of points s in $\overline{L^m}$ for which $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ has the local approximation property and G clearly contains a dense G_δ set in $\overline{L^m}$. Since one-dimensional NCCW complexes are subhomogenous algebras, we have that $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ is an approximately subhomogenous (ASH) algebra for all $s \in G$. By Proposition 8.5.1 in [7], we have that $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ is AF embeddable for all $s \in G$. □

Remark 4.4.1. The condition on the labels comes from the fact that the skew product graph is AF. In [15, Theorem 4.10], Dean scaled the action and made $\lambda_1 = 1$. In this case, the labels are all positive since the skew-product $O_n \times_c \mathbb{Z}$ is AF if and only if $c(E^1) \subseteq \mathbb{Z}^+$ or $c(E^1) \subseteq \mathbb{Z}^-$.

We can now recover Theorem 5.1 in [15].

Corollary 4.4.2. *The set of points* $(1, \lambda_2, \ldots, \lambda_n) \in (0, \infty)^{n-1}$, *for which* $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ *is an inductive limit of non-unital one-dimensional NCCW-complexes, contains a dense* G_δ *set.*

Proof. Since $\lambda_1 = 1$ we have that $L^{n-1} = (\mathbb{Q}^+)^{n-1}$ and thus $\overline{L^{n-1}} = [0, \infty)^{n-1}$. The rest follows from Theorem 4.4.1. □

Remark 4.4.2. Theorem 5.1 of [15] assumes $\lambda_1 = 1$ by rescaling. In this case, we get a continuous field over $\mathbb{R}^{n-1}$ with fibres $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$, where $\lambda = (1, \lambda_2, \ldots, \lambda_n)$. It was not necessary for us to rescale, as in our case, we can obtain a continuous field over $\mathbb{R}^n$ instead. Then, L^n will be a larger set than $(\mathbb{Q}^+)^n$.

Let $E = (E^0, E^1, r, s)$ be an infinite graph with edges $E^1 = \{e_1, e_2, \ldots\}$. We will write $L^\infty := \{(\lambda_{e_1}, \lambda_{e_2}, \ldots) \in \mathbb{Q}^\infty : \lambda$ is a labeling map on E^* and $\lambda_\mu \neq 0$ for any cycle $\mu \in E^*\}$ equipped with the product topology on $\mathbb{R}^\infty$.

Based on the results produced in the previous chapter, it is shown below that a large family of crossed products are stably projectionless.

Proposition 4.4.1. *Let* E *be a finite graph having a strongly connected component that is not a single cycle. If* $\omega : E^1 \to \mathbb{R}$ *is a labeling map with* $\omega(e) > 0$ *for all* $e \in E^1$, *then* $C^*(E) \rtimes_{\alpha^\omega} \mathbb{R}$ *is stably projectionless.*

Proof. We note that $C^*(E)$ is a simple unital C^*-algebra. Since E is a graph having a strongly connected component that is not a single cycle, there exists a $\beta > 0$, with $\rho(C_\beta) = 1$ (Proposition 3.4.1 and Proposition 3.4.2). By Proposition 3.4.3 , there exists a KMS$_\beta$ state with $\beta \neq 0$. By [35, Corollary 3.4], $C^*(E) \rtimes_{\alpha^\omega} \mathbb{R}$ is stably projectionless. □

Theorem 4.4.2. (Infinite Graph Case) *Let* E *be an row-finite graph. Then, the set of points* $\lambda \in \overline{L^\infty}$, *for which* $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ *is an inductive limit of non-unital one-dimensional NCCW-complexes, contains a dense* G_δ *set. For such* λ, *the crossed product* $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ *is AF embeddable.*

Proof. Using Proposition 4.3.1 , we can construct a sequence of finite subgraphs $F_1 \subseteq F_2 \subseteq \cdots \subseteq E$ with $E = \bigcup_{n=1}^{\infty} F_n$, so that $C^*(E)$ is an inductive limit $\varinjlim C^*(F_n)$ of finite graph algebras $C^*(F_n)$ that are invariant under α. Hence, $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R} = \varinjlim C^*(F_n) \rtimes_{\alpha^{\lambda(n)}} \mathbb{R}$. Let $\{s_k\}_{k=1}^{\infty}$ be a strictly increasing sequence with $|F_n^1| = s_n$. By Theorem 4.4.1, we have that $C^*(F_n) \rtimes_{\alpha^{\lambda(n)}} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW complexes for all $\lambda(n) \in G(n)$, where $G(n)$ contains a dense G_δ set in $\overline{L^{s_n}}$. Let $L_{s_n}^{\infty} := \{(\lambda_{e_{s_{n+1}}}, \lambda_{e_{s_{n+2}}}, \ldots) \in \mathbb{Q}^\infty : \lambda \text{ is a labeling map and } \lambda_\mu \neq 0 \text{ for any cycle } \mu \in E^*\}$. Then, $G(n) \times \overline{L_{s_n}^{\infty}}$ contains a dense G_δ set in $\overline{L^\infty}$ and so does $G := \bigcap_n G(n) \times \overline{L_{s_n}^{\infty}}$. We have that $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW complexes for all $\lambda \in G$. □

4.5 K-Theory: the Rational Fibres

The rational fibres were viewed as mapping tori over skew-graph algebras. In this section, the K-theory of these skew-graph algebras is computed. In addition, the ordered K_0 group of the skew-graph algebras $C^*(O_n \rtimes_c \mathbb{Z})$ is also calculated.

Lemma 4.5.1. *Suppose G is a countable abelian group. Then $E \times_c G \cong E \times_{-c} G$.*

Proof. Let $\phi^0 : (E \times_c G)^0 \to (E \times_{-c} G)^0$ be defined by $\phi^0(v, n) = (v, -n)$ and $\phi^1 : (E \times_c G)^1 \to (E \times_{-c} G)^1$ be defined by $\phi^0(e, n) = (e, -n)$. Then, ϕ^0 and ϕ^1 are bijective maps that satisfy $r_{E \times_{-c} G} \circ \phi^1 = \phi^0 \circ r_{E \times_c G}$ and $s_{E \times_{-c} G} \circ \phi^1 = \phi^0 \circ s^0_{E \times_c G}$. □

Proposition 4.5.1. *Let E be a finite graph without sinks and $\alpha : \mathbb{T} \curvearrowright C^*(E)$ be an action such that $\alpha_z(s_e) = z^{c(e)} s_e$, where $c : E^1 \to \mathbb{Z}$ is a labelling map with $c(E^1) \subseteq \mathbb{Z}^+$ or $c(E^1) \subseteq \mathbb{Z}^-$. If v is not a source, define $M_v := \max\{|c(e)| : r(e) = v\}$. Let $M := |E^0_{\text{sources}}| + \sum_{\{v \in E^0 : r^{-1}(v) \neq \emptyset\}} M_v$. Then, $K_0(C^*(E) \rtimes_\alpha \mathbb{T}) = \varinjlim(\mathbb{Z}^M, B)$ for some $M \times M$ matrix B and $K_1(C^*(E) \rtimes_\alpha \mathbb{T}) = 0$.*

Proof. By Proposition 4.3.1, we know that $C^*(E) \rtimes_\alpha \mathbb{T}$ is isomorphic to the graph algebra $C^*(E \times_c \mathbb{Z})$, where $E \times_c \mathbb{Z} = ((E \times_c \mathbb{Z})^0, (E \times_c \mathbb{Z})^1, r, s)$. By Lemma 4.5.1, we may assume $c(E^1) \subseteq \mathbb{Z}^+$. Let $\{s_{(e,k)}, p_{(v,k)} : k \in \mathbb{Z}, e \in E^1, v \in E^0\}$ be the canonical Cuntz-Krieger family generating $C^*(E \times_c \mathbb{Z})$. Since the skew product has no cycles, $C^*(E \times_c \mathbb{Z})$ is AF.

Let $V_m = \{(v, k) : v \in E^0, k \in \mathbb{Z} \text{ and } -\infty \leq k \leq m\}$ for $m \geq 0$. For $m \geq 1$, define F_m to be the subgraph of $E \times_c \mathbb{Z}$ with vertices

$$F_m^0 = V_m \cup \{(v, m+1), \ldots, (v, m-1+M_v) : v \in E^0, r^{-1}(v) \neq \emptyset\}$$

and edges

$$F_m^1 := s^{-1}(V_{m-1}).$$

We have that each F_m is a graph without loops, where $F_m \subseteq F_{m+1}$ for $m \geq 1$ and $E \times_c \mathbb{Z} = \bigcup_{n=1}^{\infty} F_m$. Let A_m denote the C^*-subalgebra of $C^*(E \times_c \mathbb{Z})$ generated by $\{s_{(e,\ell)} : (e, \ell) \in F_m^1\} \cup \{p_{(v,n)} : (v, n) \in F_m^0\}$. The generating set for A_m is a Cuntz-Krieger F_m family in $C^*(E \times_c \mathbb{Z})$ with all projections nonzero. Hence, by the Cuntz-Krieger uniqueness theorem, there is an injection of $C^*(F_m)$ into $C^*(E \times_c \mathbb{Z})$ and this map gives $C^*(F_m) \cong A_m$. Thus, $C^*(F_m) \subseteq C^*(F_{m+1})$ and $C^*(E \times_c \mathbb{Z}) = \overline{\bigcup_{m=1}^{\infty} C^*(F_m)}$.

A typical element in the spanning set for $C^*(F_m)$ is $s_\mu s_\nu^*$ with $r(\mu) = r(\nu)$. Suppose $r(\mu) = r(\nu) = (w, k)$. If (w, k) is not a sink, we can apply the Cuntz-Krieger relations, so that $s_\mu s_\nu^*$ can be written as a finite sum of terms of the form $s_\alpha s_\beta^*$, where $r(\alpha) = r(\beta)$ is a sink. The set of all sinks in the graph F_m is the set $S_{F_m} := \{(v, m) : v \in E^0\} \cup \{(v, m+1), \ldots, (v, m-1+M_v) : v \in E^0, r^{-1}(v) \neq \emptyset\}$. Therefore,

$$C^*(F_m) = \overline{\text{span}}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in S_{F_m}\}.$$

Fix an element $(v,k) \in S_{F_m}$. Let

$$F_m^{(v,k)} = \{\alpha \in F_m^* : r(\alpha) = (v,k)\}$$

and

$$A_{(v,k)} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F_m^{(v,k)}\}.$$

The elements $s_\alpha s_\beta^*$ with $\alpha, \beta \in F_m^{(v,k)}$ form a family of matrix units, and thus, $A_{(v,k)}$ is isomorphic to the algebra of compact operators on $\ell^2(F_m^{(v,k)})$. For any two elements $(v,k),(w,n) \in S_{F_m}$, $A_{(v,k)}$ is orthogonal to $A_{(w,n)}$ when $(v,k) \neq (w,n)$. Hence,

$$C^*(F_m) = \bigoplus_{(v,k)\in S_{F_m}} A_{(v,k)}.$$

Since $p_{(v,k)}$ is a rank one projection in $A_{(v,k)}$, we have that $K_0(A_{(v,k)})$ is a free abelian group generated by $[p_{(v,k)}]$. Therefore,

$$K_0(C^*(F_m)) = K_0\left(\bigoplus_{(v,k)\in S_{F_m}} A_{(v,k)}\right) = \bigoplus_{(v,k)\in S_{F_m}} K_0(A_{(v,k)}) = \bigoplus_{(v,k)\in S_{F_m}} \mathbb{Z}[p_{(v,k)}].$$

Continuity of K_0 gives $K_0(C^*(E \times_c \mathbb{Z})) = \varinjlim K_0(C^*(F_m))$.

To calculate the bonding maps $\phi_{m,m+1} : K_0(C^*(F_m)) \longrightarrow K_0(C^*(F_{m+1}))$, we will see how the projections $[p_{(v,k)}]$, with $(v,k) \in S_{F_m}$, decompose in $K_0(C^*(F_{m+1}))$. If $v \in E^0$, then

$$\begin{aligned}
[p_{(v,m)}] &= \sum_{s(e,n)=(v,m)} [s_{(e,n)} s_{(e,n)}^*] \\
&= \sum_{s(e,n)=(v,m)} [p_{r(e,n)}] \\
&= \sum_{s(e)=v} [p_{(r(e),m+c(e))}],
\end{aligned}$$

where $m < m + c(e) \leq m + M_{r(e)}$.

Since $S_{F_m} \cap S_{F_{m+1}} = S_{F_m} \backslash \{(v, m) : v \in E^0\}$, we have that

$$\phi_{m,m+1}([p_{(v,k)}]) = \begin{cases} [p_{(v,k)}] & \text{if } (v,k) \in S_{F_m} \backslash \{(v,m) : v \in E^0\} \\ \sum_{s(e)=v} [p_{(r(e),m+c(e))}] & \text{otherwise.} \end{cases} \tag{4.3}$$

We note that

$$\begin{aligned} |S_{F_m}| &= |E^0| + |F_m^0 \backslash V_m| \\ &= |E^0| + \sum_{\{v \in E^0 : r^{-1}(v) \neq \emptyset\}} (M_v - 1) \\ &= |E^0| + \sum_{\{v \in E^0 : r^{-1}(v) \neq \emptyset\}} M_v - |E^0 \backslash E^0_{\text{sources}}| \\ &= |E^0_{\text{sources}}| + \sum_{\{v \in E^0 : r^{-1}(v) \neq \emptyset\}} M_v \\ &= M. \end{aligned}$$

The matrix representations of the bonding maps $\phi_{m,m+1}$ are all the same and we will denote them by B. Hence, $K_0(C^*(E \times_c \mathbb{Z})) \cong \varinjlim(\mathbb{Z}^M, B)$. □

As a consequence of Proposition 4.5.1, we can now recover the result for the gauge action (see Corollary 7.14 of [46]).

Corollary 4.5.1. *Let $C^*(E)$ be a row-finite graph without sinks and let $\gamma : \mathbb{T} \curvearrowright C^*(E)$ be the standard gauge action. Then, $K_0(C^*(E) \rtimes_\gamma \mathbb{T}) = \varinjlim(\mathbb{Z}^{E^0}, A_E^t)$, where A_E is the vertex matrix.*

Proof. For the standard gauge action, we have $c(e) = 1$ for all edges $e \in E^1$. Thus, $C^*(E) \rtimes_\gamma \mathbb{T} \cong C^*(E \times_1 \mathbb{Z})$. Using Proposition 4.5.1, we see that $F_m^0 = V_m$, $F_m^1 := s^{-1}(V_{m-1})$ and $S_{F_m} := \{(v, m) : v \in E^0\}$. We note $M = |E^0|$ and for all $v \in E^0$, we have that

$$\begin{aligned}\phi_{m,m+1}([p_{(v,m)}]) &= \sum_{s(e)=v} [p_{(r(e),m+1))}] \\ &= \sum_{w\in E^0} A_E(v,w)[p_{(w,m+1))}].\end{aligned}$$

Hence, in this case, the bonding map is multiplication by the transpose of the vertex matrix, as required. □

4.5.1 The Cuntz Algebra Case

Let O_n be the Cuntz algebra with corresponding graph having vertex v and edges $\{e_i\}_{i=1}^n$. The skew-product graph algebra $C^*(O_n \times_c \mathbb{Z})$ is AF if and only if $c(E^1) \subseteq \mathbb{Z}^+$ or $c(E^1) \subseteq \mathbb{Z}^-$. Without loss of generality, we assume $c(E^1) \subseteq \mathbb{Z}^+$. Suppose we have $s \leq n$ distinct labels; namely, $k_1 < k_2 < \cdots < k_s$.

Here, we note that $k_s = M$. For all $j = 1, 2, \ldots, k_s$, define

$$c_j := \begin{cases} |\{e \in E^1 : c(e) = j\}| & \text{if } j \in \{k_1, k_2, \ldots, k_s\} \\ 0 & \text{otherwise.} \end{cases}$$

Using Proposition 4.5.1, we see that $S_{F_m} = \{(v,m), (v,m+1), \ldots, (v, m-1+M_v)\}$ and the bonding maps $\phi_{m,m+1}$ send

$$\begin{aligned}[p_{(v,m)}] \longmapsto &\sum_{s(e)=v} [p_{(r(e),m+c(e))}] \\ =&\sum_{i=1}^{n} [p_{(v,m+c(e_i))}] \\ =&\sum_{i=1}^{s} c_{k_i}[p_{(v,m+k_i)}],\end{aligned}$$

while the remaining elements remain fixed under $\phi_{m,m+1}$. Hence,

$$B = \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k_s-1} & 0 & 0 & \cdots & 1 \\ c_{k_s} & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4.4}$$

is the matrix representation of the bonding maps $\phi_{m,m+1}$. Therefore, we have that $K_0(C^*(O_n \times_c \mathbb{Z})) \cong \varinjlim(\mathbb{Z}^{M_v}, B)$.

Also, the determinant of B is $c_{k_s}(-1)^{k_s+1} \neq 0$. So, if we suppose that $c_{k_s} = 1$, then the bonding maps are bijective and in this case, $K_0(C^*(O_n \times_c \mathbb{Z})) \cong \mathbb{Z}^{M_v}$.

The bonding maps in Proposition 4.5.2 are nonnegative unimodular primitive matrices in $\mathrm{M}_k(\mathbb{Z})$. A matrix A is *unimodular* if the determinant of A is $+1$ or -1 and *primitive* if A is nonnegative and $A^m > 0$ for some positive integer m. The result below will be useful for calculating the positive cone of the K_0 group.

Proposition 4.5.2. [54, p. 464] *Suppose we are given a sequence*

$$\mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \cdots$$

where A is a nonnegative unimodular primitive matrix in $\mathrm{M}_k(\mathbb{Z})$. Then, the resulting stationary dimension group $\varinjlim(\mathbb{Z}^k, A)$ has a unique state, and we can express its positive cone as

$$P_{(1,\alpha_2,\ldots,\alpha_n)} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^k : x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n > 0\} \cup \{(0, \ldots, 0)\},$$

where $(1, \alpha_2, \ldots, \alpha_n)$ is the eigenvector of the Perron-Frobenius eigenvalue of A^{tr}, at least one of α_i is irrational, and $\alpha_2, \ldots, \alpha_n > 0$.

For the rest of the chapter, we will suppose that $s = n$ and $\gcd(k_1, \ldots, k_n) = 1$. Then, $c_{k_1} = c_{k_2} = \cdots c_{k_n} = 1$ and $c_j = 0$ otherwise. We will use the notation $O_n(k_1, k_2, \ldots, k_n)$ to represent $C^*(O_n \times_c \mathbb{Z})$, where c is a labeling map with distinct labels $k_n > k_{n-1} > \cdots > k_1 > 0$. We denote the transpose of the matrix B in (4.4) as

$$A_{(k_1,\ldots,k_n)} = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{k_n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{4.5}$$

This matrix is known in the literature as the Leslie matrix. The matrix $A_{(k_1,\ldots,k_n)}$ has characteristic polynomial $x^{k_n} - x^{k_n-k_1} - \cdots - x^{k_n-k_{n-1}} - 1$.

In order to apply Proposition 4.5.2, a description of the Perron-Frobenius eigenvalue and its corresponding eigenvector for $A_{(k_1,k_2,\ldots,k_n)}$ is needed. This is described in Lemma 4.5.2, along with the fact that α is the limit of a ratio of terms from a difference equation. The description of α in this way was given in [24], but with more cumbersome calculations as the size of the matrix increased (for example, [24, p. 26]). The proof of Lemma 4.5.2 makes use of some standard results from matrix theory.

Lemma 4.5.2. *The matrix $A_{(k_1,k_2,\ldots,k_n)}$ has eigenvector $(1, \alpha^{-1}, \ldots, \alpha^{-k_n+1})^{tr}$, where α is the Perron-Frobenius eigenvalue of $A_{(k_1,k_2,\ldots,k_n)}$ and is irrational. It satisfies $\alpha = \lim_{m\to\infty} \dfrac{f_{m+k_n}}{f_{m+k_n-1}}$, where*

$$f_{m+k_n} = f_{m+k_n-k_1} + f_{m+k_n-k_2} + \cdots + f_m$$

is a difference equation with initial conditions $f_0 = f_1 = \cdots = f_{k_n-2} = 0$ and

$f_{k_n-1} = 1$.

Proof. The matrix $A_{(k_1,k_2,\ldots,k_n)}$ yields a difference equation of the form

$$f_{m+1} = f_{m-k_1+1} + f_{m-k_2+1} + \cdots + f_{m-k_n+1},$$

with initial conditions $f_0 = f_1 = \cdots = f_{k_n-2} = 0$ and $f_{k_n-1} = 1$ or equivalently, in matrix form

$$\begin{pmatrix} f_{m+k_n} \\ \vdots \\ f_{m+2} \\ f_{m+1} \end{pmatrix} = A_{(k_1,k_2,\ldots,k_n)} \begin{pmatrix} f_{m+k_n-1} \\ \vdots \\ f_{m+1} \\ f_m \end{pmatrix}, \qquad \begin{pmatrix} f_{k_n-1} \\ f_{k_n-2} \\ \vdots \\ f_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{(see [42, pp.683–684]).}$$

If we let $\mathbf{g}(m) = \begin{pmatrix} f_{m+k_n-1} \\ \vdots \\ f_{m+1} \\ f_m \end{pmatrix}$, then it is not hard to see that $\mathbf{g}(m) = A^m_{(k_1,k_2,\ldots,k_n)}\mathbf{g}(0)$.

The matrix $A_{(k_1,k_2,\ldots,k_n)}$ is primitive if and only if the $\gcd(k_1, \ldots, k_n) = 1$ (see, for example, Theroem 6.11 in [11]). If $r = \rho(A_{(k_1,k_2,\ldots,k_n)})$, then $\lim_{m\to\infty} \left(\frac{A_{(k_1,k_2,\ldots,k_n)}}{r}\right)^m = \frac{\mathbf{p}\mathbf{q}^{tr}}{\mathbf{q}^{tr}\mathbf{p}} > 0$, where $\mathbf{p}$ and $\mathbf{q}$ are the Perron-Frobenius eigenvectors of $A_{(k_1,k_2,\ldots,k_n)}$ and $A^{tr}_{(k_1,k_2,\ldots,k_n)}$, respectively [42, p. 674]. From this, we get that

$$\lim_{m\to\infty} \frac{\mathbf{g}(m)}{||\mathbf{g}(m)||_1} = \mathbf{p},$$

where $\mathbf{p}$ is the Perron-Frobenius eigenvector of $A_{(k_1,k_2,\ldots,k_n)}$ (see [42, p.684]). For $0 \leq q \leq k_n - 1$, $\lim_{m\to\infty} \frac{f_{m+q}}{||\mathbf{g}(m)||_1}$ exists and is positive. Hence, so is $\lim_{m\to\infty} \frac{f_{m+q}}{f_{m+k_n-1}}$.

Since

$$\frac{f_{m+k_n}}{f_{m+k_n-1}} = \frac{f_{m+k_n-k_1}}{f_{m+k_n-1}} + \frac{f_{m+k_n-k_2}}{f_{m+k_n-1}} + \cdots + \frac{f_m}{f_{m+k_n-1}}, \tag{4.6}$$

$\lim_{m\to\infty} \frac{f_{m+k_n}}{f_{m+k_n-1}}$ exists and we will denote it by α.

Taking the limit of both sides of equation (4.6), we get $\alpha = \alpha^{-(k_1-1)} + \alpha^{-(k_2-1)} + \cdots + \alpha^{-(k_n-1)}$, or equivalently,

$$\alpha^{k_n} - \alpha^{k_n-k_1} - \alpha^{k_n-k_2} - \cdots - 1 = 0.$$

Therefore, α satisfies the characteristic polynomial of $A_{(k_1,k_2,\ldots,k_n)}$. By Descartes rule of signs, the characteristic polynomial has one positive root and since α is positive, it must be the Perron-Frobenius eigenvalue.

Lastly, we have that

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{k_n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha^{-1} \\ \alpha^{-2} \\ \vdots \\ \alpha^{-k_n+1} \end{pmatrix} = \begin{pmatrix} \alpha^{-k_1+1} + \alpha^{-k_2+1} + \cdots + \alpha^{-k_n+1} \\ 1 \\ \alpha^{-1} \\ \vdots \\ \alpha^{-k_n} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 1 \\ \alpha^{-1} \\ \alpha^{-2} \\ \vdots \\ \alpha^{-k_n+1} \end{pmatrix}.$$

The only possible rational root of the characteristic polynomial is -1, hence α must be irrational. $\square$

Theorem 4.5.1. *$K_0(O_n(k_1,\ldots,k_n)) = \mathbb{Z}^{k_n}$ and $K_0^+(O_n(k_1,\ldots,k_n)) = P_{(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})}$, where*

$$P_{(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})} = \{(x_1,x_2,\ldots,x_{k_n}) \in \mathbb{Z}^{k_n} : x_1 + \alpha^{-1}x_2 + \cdots + \alpha^{-k_n+1}x_{k_n} > 0\} \cup \{(0,0,\ldots,0)\},$$

and α is the Perron-Frobenius eigenvalue of $A_{(k_1,k_2,\ldots,k_n)}$ which satisfies $\alpha = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ with α irrational. Furthermore, if the characteristic polynomial of $A_{(k_1,k_2,\ldots,k_n)}$ is irreducible over the rationals, then

$$\left(K_0(O_n(k_1,k_2,\ldots,k_n)), K_0^+(O_n(k_1,k_2,\ldots,k_n)), \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^{tr}\right)$$
$$\cong (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}, (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}) \cap \mathbb{R}_+, 1).$$

Proof. By Proposition 4.5.1, Lemma 4.5.2 and Proposition 4.5.2 ([54, p. 464]), we have the K_0 group and its cone are described as above. The map $(1,\alpha^{-1},\ldots,\alpha^{-k_n+1}) : \mathbb{Z}^{k_n} \to \mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}$ is a positive surjective homomorphism that preserves the order unit and the image of the cone $K_0^+(O_n(k_1,k_2,\ldots,k_n))$ is exactly $(\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}) \cap \mathbb{R}_+$. Furthermore, if the characteristic polynomial is irreducible, then the map $(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})$ is injective since the set $\{1,\alpha^{-1},\ldots,\alpha^{-k_n+1}\}$ is linearly independent. Indeed, the set $\{1,\alpha,\ldots,\alpha^{k_n-1}\}$ is linearly independent since the characteristic polynomial is irreducible and therefore, so is the set $\{1,\alpha^{-1},\ldots,\alpha^{-k_n+1}\}$. Hence, we have an order isomorphism, as required. □

Remark 4.5.1. $K_0(O_n(k_1,k_2,\ldots,k_n))$ is not totally ordered when the number of nonzero even entries is one greater than the number of nonzero odd entries in the first row of the bonding maps $A_{(k_1,k_2\ldots,k_n)}$, since the characteristic polynomial will have -1 as a root (see [27, pp.63–64]).

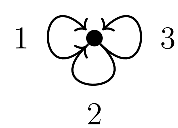
The K-theory of the standard Fibonacci algebra was calculated in [13] and extended for the generalized Fibonacci algebras in [24]. The standard embedding was given by the matrix $A_{(k_1,k_2,\ldots,k_n)}$, where $k_j = j$ for $j = 1, \ldots, n$. As a consequence of Theorem 4.5.1, we arrive at the same results, but in a more indirect way.

Corollary 4.5.2. *Suppose $k_j = j$ for $j = 1, 2, \ldots, n$. Then,*

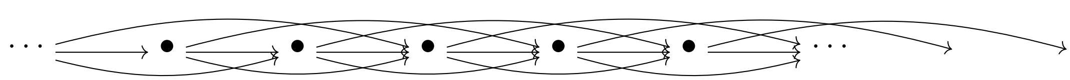
$$\left(K_0(O_n(k_1, k_2, \ldots, k_n)), K_0^+(O_n(k_1, k_2, \ldots, k_n)), \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^{tr}\right)$$
$$\cong (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}, (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}) \cap \mathbb{R}_+, 1).$$

Proof. By [6, Theorem 2], the characteristic polynomial of $A_{(k_1,k_2\ldots,k_n)}$ is irreducible over the rationals. Then, the result follows from Theorem 4.5.1.

Example 4.5.1. Suppose we have the Cuntz-algebra O_3 with the following labels :



Then, we have the following skew-product graph $O_3(1, 2, 3)$:



Hence, $K_0(O_3(1, 2, 3)) = \varinjlim \left(\mathbb{Z}^3, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right) \cong \mathbb{Z}^3$. In this case, $\alpha \approx 1.8392$ is the Perron-Frobenus eigenvalue of the above bonding map and

$$\left(K_0(O_3(1, 2, 3)), K_0^+(O_3(1, 2, 3)), \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^{tr}\right)$$
$$\cong (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \alpha^{-2}\mathbb{Z}, (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \alpha^{-2}\mathbb{Z}) \cap \mathbb{R}_+, 1).$$

Chapter 5

Conclusion and Discussion

This thesis dealt with the study of C^*-dynamical systems consisting of quasi-free actions on graph algebras. First and foremost, the main results of this thesis stemmed from the characterization of KMS states for quasi-free actions on finite-graph algebra in terms of a polyhedral set in $\mathbb{R}^{E^0}$. Since every Toeplitz algebra is a graph algebra (whose corresponding graph can be easily described using Definition 3.2.1), the results were recovered to suit this point of view as well.

Given $\beta \geq 0$ and a quasi-free action α^ω corresponding to a labeling map ω, the result in Theorem 3.3.1 shows that the simplex of all KMS_β states is affine-isomorphic to the polyhedral set $L_{\beta,\alpha^\omega} = \{x = (x_v) \in \mathbb{R}^{E^0} : R_\beta x = x\}$, for a certain matrix R_β (Remark 3.3.2). More precisely, for a fixed $\beta \geq 0$, a solution x to the matrix equation $R_\beta x = x$, gives rise to a KMS_β state satisfying

$$\phi_m(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} m_{r(\mu)}.$$

Hence the vectors in the set L_{β,α^ω} determine the construction of the KMS states. This characterization gave a method by which the KMS states for quasi-free actions on finite-graph algebras can be described, while extending the results known for the gauge action on a Toeplitz algebra.

The result in Corollary 3.4.1 shows the existence of a KMS state at a critical inverse temperature $\beta = \beta_c$ and in Theorem 3.5.1, a precise description of the simplex of KMS$_\beta$ states is given for $\beta > \beta_c$. Precisely, the simplex of KMS states is affine-isomorphic to

$$\sum\nolimits_\beta := \{\epsilon \in [0,\infty)^{E^0} : \epsilon \cdot y = 1 \text{ and } \epsilon_v = 0 \text{ for } v \in E^0_{\text{reg}}\}.$$

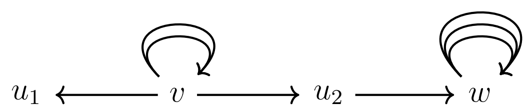
Theorem 3.6.1 looked at the behaviour of the KMS states of the C^*-dynamical system $(C^*(E), \alpha^\omega)$ when $E\backslash H$ is strongly connected, where H is the set of sinks. In this case, a complete description of all KMS is given: for $\beta > \beta_c$, the set of KMS$_\beta$ states is a simplex of dimension $|H| - 1$ and if $\beta = \beta_c$, the system has a unique KMS$_{\beta_c}$ state which factors through a KMS$_{\beta_c}$ state of $(C^*(E\backslash H), \alpha^\omega)$. The state that factors through is the only KMS state of the C^*-dynamical system $(C^*(E\backslash H), \alpha^\omega)$. If $\beta < \beta_c$, the C^*-dynamical system $(C^*(E), \alpha^\omega)$ has no KMS states.

In their paper [28], an Huef, Laca, Raeburn, and Sims studied KMS states for graphs having a reducible vertex matrix; in other words, when the graph is not strongly connected. They looked at the strongly connected components of a graph and their interactions to give general results about the KMS states for the gauge actions of the reals acting on the Toeplitz algebra. They applied their results to many examples and were able to compute all KMS states on the Toeplitz algebra $\mathcal{T}C^*(E)$ and the graph algebra $C^*(E)$.

The characterization given in this thesis gives a practical method of constructing the KMS states in the quasi-free setting. Using a result from Tomforde-Muhly (Proposition 3.2.1), the Toeplitz algebra $\mathcal{T}C^*(E)$ is the graph algebra $C^*(E_\mathcal{T})$, where $E_\mathcal{T}$ is a graph constructed from E (Definition 3.2.1). By computing the KMS states on $C^*(E_\mathcal{T})$, the KMS states on $C^*(E)$ can be seen by looking at the components of the corresponding vector in $\mathbb{R}^{E^0_\mathcal{T}}$. In particular, if ϕ_m is a KMS state of $(C^*(E_\mathcal{T}), \alpha^\omega)$ that depends on a vector in $\mathbb{R}^{E^0_\mathcal{T}}$, then ϕ_m factors through a KMS state of $(C^*(E), \alpha^\omega)$

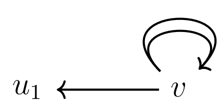
if $m_v = 0$ for each $v \in E^0_{\mathcal{T}} \backslash E^0$.

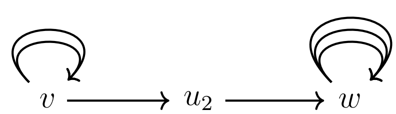
The techniques from this thesis allow for a good foundation in extending the results of [28] to quasi-free actions. It would be interesting to see how the subgraphs of a given graph help determine the KMS states of the C^*-dynamical system. In Example 3.9.1, the KMS states were computed for the gauge action of the reals, where E is the graph seen below:



For $\beta = \ln 3$, we have a 2-dimensional simplex of KMS_β states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 \end{pmatrix}^{tr}$ and
$m_2^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}^{tr}$.

Therefore, the following subgraphs will help determine the KMS state of $(C^*(E), \gamma)$:





We notice that the extreme points of the $\mathrm{KMS}_{\ln 3}$ simplex of $(C^*(E), \gamma)$ reduces to finding the $\mathrm{KMS}_{\ln 3}$ states of the C^*-dynamical systems corresponding to each subgraph above. Each will give us a unique $\mathrm{KMS}_{\ln 3}$ state that corresponds to an extreme point of the 2-dimensional simplex. In general, if given a graph E and β, is it possible to describe subgraphs of E that determine the extreme points of a KMS_β simplex? This may give some insight into describing a graph from a specified KMS_β simplex.

Chapter 4 of this thesis analyzed the structure of crossed products by quasi-free actions; in particular, the construction of crossed products as inductive limits of non-unital one-dimensional NCCW complexes. The main results of this chapter extended Dean's results in [15] to crossed products of more general graph algebras. The case of finite graphs was established in Theorem 4.4.1 and the case which allowed infinite graphs was established in Theorem 4.4.2:

Theorem 4.4.1 (Finite Graph Case) *Let E be a finite graph with edges $E^1 = \{e_1, e_2, \ldots, e_m\}$. Then, the set of points $\lambda \in \overline{L^m}$, for which $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes, contains a dense G_δ set in $\overline{L^m}$. For such λ, the crossed product $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable.*

Theorem 4.4.2 (Infinite Graph Case) *Let E be a row-finite graph. Then, the set of points $\lambda \in \overline{L^\infty}$, for which $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes, contains a dense G_δ set in $\overline{L^m}$. For such λ, the crossed product $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable.*

The main idea of the argument followed from [15], but the use of some graph algebra machinery enabled some of the proofs to be more concise. First, we view the crossed product as a continuous field of $\mathbb{R}^{E^1}$ and then analyze the structure of the 'rational fibres'; the 'rational' fibres are crossed products by periodic quasi-free actions and these can be regarded as mapping tori over graph algebras. Each mapping torus was decomposed as an inductive limit of non-unital one-dimensional NCCW complexes (Proposition 4.3.3). Finally, some approximation theorems were used to find that, for a dense G_δ set, the crossed products can be written as an inductive limit of non-unital one-dimensional NCCW complexes. The graph algebras used in the construction were skew-product graphs and the K-theory was calculated for finite graphs with no sinks (Proposition 4.5.1). This proposition was applied to the Cuntz algebra case and the ordered K_0 group was calculated in Theorem 4.5.1:

Theorem 4.5.1 $K_0(O_n(k_1, \ldots, k_n)) = \mathbb{Z}^{k_n}$ and

$$K_0^+(O_n(k_1,\ldots,k_n)) = P_{(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})},$$

where

$$\begin{aligned} P_{(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})} &= \{(x_1,x_2,\ldots,x_{k_n}) \in \mathbb{Z}^{k_n} : x_1 + \alpha^{-1}x_2 + \cdots + \alpha^{-k_n+1}x_{k_n} > 0\} \\ &\cup \{(0,0,\ldots,0)\}, \end{aligned}$$

and α is the Perron eigenvalue of $A_{(k_1,k_2,\ldots,k_n)}$ that satisfies $\alpha = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ with α irrational. Furthermore, if the characteristic polynomial of $A_{(k_1,k_2,\ldots,k_n)}$ is irreducible over the rationals, then

$$\begin{aligned} &\left(K_0(O_n(k_1,k_2,\ldots,k_n)), K_0^+(O_n(k_1,k_2,\ldots,k_n)), \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^{tr}\right) \\ &\cong (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}, (\mathbb{Z} + \alpha^{-1}\mathbb{Z} + \cdots + \alpha^{-k_n+1}\mathbb{Z}) \cap \mathbb{R}_+, 1). \end{aligned}$$

The above theorems add some useful insight into the structure and properties of crossed products of graph algebras by quasi-free actions and pose some interesting questions.

The results in Chapter 4 show that in situations where the graph algebra is not a Cuntz algebra, the crossed product can still embed into an AF algebra. There are no restrictions on the graph; however, in this case, there is no specificity with respect to the labels. Namely, we only know that crossed products are AF-embeddable for some dense G_δ set of labels.

In the Cuntz algebra case, Katsura showed that if $\omega(e_i)$ is not in the closed subsemigroup generated by the set $\omega(e_1), \omega(e_2), \ldots, \omega(e_n)$ for any $i \in \{1,2,\ldots,n\}$, then crossed product $O_n \rtimes_{\alpha^\omega} \mathbb{R}$ is AF-embeddable [32]. Katsura's method included some technical lemmas in order to construct a suitable AF-algebra. These techniques were applied in [23] to extend the AF-embeddability result to a larger family of graphs, but the class of graphs had restrictions. It would be nice to give a more precise description of the labels for the situations of AF-embeddability that includes a larger family of graphs.

It would also be interesting to explore the properties of simple crossed products of graph algebras by quasi-free actions and provide some insight into the following question: can we extend the following theorem proven by Kishimoto and Kumjian in [36]?

Theorem 5.0.1. [36, Theorem 2] *Suppose that* $\alpha_t(S_k) = e^{i\lambda_k t} S_k$ *for* $k = 1, \ldots, n$ *is a quasi-free action of* $\mathbb{R}$ *on the Cuntz algebra* O_n*. If the closed subsemigroup generated by all* λ_k *is* $\mathbb{R}$*, then* $O_n \rtimes_\alpha \mathbb{R}$ *is a stable, purely infinite, simple* C^**-algebra.*

In [18], the ideal structure of crossed products of graph algebras by quasi-free actions was investigated and necessary and sufficient conditions were given for the simplicity of these crossed products. The notion of a labeling map ω being simple is used to describe the simplicity of crossed products and is defined below:

Definition 5.0.1. [18, Definition 3.1] Let $E = (E^0, E^1, r, s)$ be a row finite graph without sinks, G be a locally compact abelian group with dual group Γ and $\omega : E^* = \bigcup_{n \geq 0} E^n \to \Gamma$ be a labeling map. A family of sets $X = \{X_v : v \in E^0\}$, where X_v is a closed subset of Γ, will be called an E-class of subsets of Γ. An E-class X of subsets of Γ will be called ω-hereditary if, for any $\mu \in E^*$, $X_{r(\mu)} + \omega_\mu \subseteq X_{s(\mu)}$; it will be called ω-saturated if, for any $v \in E^0$, $X_v \subseteq \bigcup_{i=1}^{n} (\omega_{e_i} + X_{r(e_i)})$, where $e_1, e_2, \ldots, e_n (n \geq 1)$ are all edges which v emits; and it will be trivial if either $X_v = \Gamma$(for all $v \in E^0$) or $X_v = \emptyset$ (for all $v \in E^0$). We say that a labeling map ω is simple if any ω-hereditary and ω-saturated E-class $X = \{X_v : v \in E^0\}$ of subset of Γ is trivial, i.e., either $X_v = \Gamma$ for any $v \in E^0$, or $X_v = \emptyset$ of any $v \in E^0$.

The following result, which was proven in [18], describes the simplicity of the crossed products in terms of the labeling map:

Theorem 5.0.2. [18, Theorem 4.4] *Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph without sinks, let G be a locally compact abelian group with dual group Γ, and let $\omega : E^* = \bigcup_{n\geq 0} E^n \to \Gamma$. Then, $C^*(E) \rtimes \mathbb{R}$ is simple if and only if ω is simple and every loop in E has an exit.*

Since the Cuntz algebra gives a strongly connected graph, a natural place to start is when E is strongly connected graph. Here are some simple results that give a situation of simplicity based on the labels in the graph.

Lemma 5.0.1. *Let E be a strongly connected finite graph and Γ_v be the closed subsemigroup generated by w_μ, for all paths μ from v to v. If $\Gamma_v = \mathbb{R}$, for some $v \in E^0$, then $\Gamma_v = \Gamma_w$ for all w in E^0.*

Proof. Choose a path α from v to w and β from w to v. We note that $\eta = \alpha\beta$ is a path from v to v and $\eta' = \beta\alpha$ is a path from w to w with $\omega_\eta = \omega_{\eta'}$. Then, $\Gamma_v + \omega_\eta \subseteq \Gamma_w$ and hence, $\Gamma_w = \mathbb{R}$. □

Proposition 5.0.1. *Let E be a strongly connected graph that is not a single cycle and $\Gamma_v = \mathbb{R}$, for some $v \in E^0$. Then, $C^*(E) \rtimes_\alpha \mathbb{R}$ is simple.*

Proof. Let $X = \{X_v\}$ be a ω-hereditary and ω-saturated E-class of subsets of $\mathbb{R}$, with $X_{v_0} \neq \emptyset$ and let $\gamma \in X_{v_0}$. Since X is ω-hereditary, $\gamma + \omega_\mu \in X_{v_0}$ for all paths μ from v_0 to v_0 and $\Gamma_{v_0} + \gamma \subseteq X_{v_0}$. By the previous lemma, $\Gamma_v = \mathbb{R}$ for all $v \in E^0$; therefore, $X_{v_0} = \mathbb{R}$. Hence, ω is simple and by Theorem 4.4 in [18], we have that $C^*(E) \rtimes_{\alpha^\omega} \mathbb{R}$ is simple. □

If Proposition 5.0.1 is reduced to the Cuntz algebra, then the crossed product will be purely infinite (Theorem 5.0.2). It would be interesting to see if a similar phenomena occurs for strongly connected graphs.

References

[1] T. Bates, J. Hong, I. Raeburn, and W. Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois Journal of Math. **46** (2002), no. 4, 1159–1176. MR 1988256 (2004i:46105)

[2] T. Bates, D. Pask, I. Raeburn, and W. Szymański, *The C^*-algebras of row-finite graphs*, New York Journal of Math. **6** (2000), 307–324 (electronic). MR 1777234 (2001k:46084)

[3] M.S. Bazaraa, H.D. Sherali, and C.M. Shetty, *Nonlinear programming: theory and algorithms*, third ed., John Wiley & Sons, 2006.

[4] B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Springer, Berlin, 2006.

[5] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical mechanics*, vol. 2, Springer-Verlag, New York, 1997.

[6] A. Brauer, *On algebraic equations with all but one root in the interior of the unit circle*, Math. Nachr. **4** (1951), 250–257. MR 0041975 (13,32c)

[7] N.P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, vol. 88, American Mathematical Soc., Providence, R.I, 2008.

[8] T.M. Carlsen and N.S. Larsen, *Partial actions and KMS states on relative graph C^*-algebras*, arXiv e-prints (2013), 1–33.

[9] C. Chlebovec, *KMS states for quasi-free actions on finite-graph algebras*, Journal of Operator Theory **75** (2016), no. 1, 119–138.

[10] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*, vol. 55, American Mathematical Soc., 2008.

[11] P. Cull, M. Flahive, and R. Robson, *Difference equations: from rabbits to chaos*, Springer Science & Business Media, 2005.

[12] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Inventiones Mathematicae **56** (1980), no. 3, 251–268.

[13] K. Davidson, *C^*-algebras by example*, American Mathematical Society, Providence, R.I, 1996.

[14] G.G. de Castro and F. de L. Mortari, *KMS states for the generalized gauge action on graph algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **36** (2014), no. 4, 114–128. MR 3306919

[15] A. Dean, *A continuous field of projectionless C^*-algebras*, Canadian Journal of Mathematics **53** (2001), no. 1, 51–72. MR 1814965 (2002a:46087)

[16] E. Effros, *Dimensions and C^*-algebras*, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I, 1981.

[17] S. Eilers, T.A. Loring, and G.K. Pedersen, *Stability of anticommutation relations: an application of noncommutative CW complexes*, J. Reine Angew. Math. **499** (1998), 101–143. MR 1631120 (99e:46067)

[18] G.A. Elliott and X. Fang, *Ideals and simplicity of crossed products of graph C^*-algebras by quasi-free actions*, Münster Journal of Math. **3** (2010), 11–28.

[19] M. Enomoto, M. Fujii, and Y. Watatani, *KMS states for gauge action on O_A*, Math. Japon. **29** (1984), no. 4, 607–619. MR 759450 (86b:46110)

[20] M. Enomoto and Y. Watatani, *A graph theory for C^*-algebras*, Math. Japon **25** (1980), no. 4, 435–442.

[21] D.E. Evans, *On O_n*, RIMS, Kyoto Univ. **16** (1980), 915–927.

[22] R. Exel and M. Laca, *Partial dynamical systems and the KMS condition*, Comm. Math. Phys. **232** (2003), no. 2, 223–277. MR 1953065 (2006k:46112)

[23] X. Fang, *AF-embeddability of crossed products of certain graph C^*-algebras by quasi-free actions*, C. R. Math. Rep. Acad. Sci. Canada **31** (2009), 76–86.

[24] C.B. Flournoy Jr, *N-parameter Fibonacci AF C^*-algebras*, Ph.D. thesis, University of Iowa, 2011.

[25] N.J. Fowler, M. Laca, and I. Raeburn, *The C^*-algebras of infinite graphs*, Proceedings of the American Mathematical Society (2000), 2319–2327.

[26] N.J. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. Journal **48** (1999), no. 1, 155–181. MR 1722197 (2001b:46093)

[27] D. Handelman, *Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains*, Journal of Operator Theory **6** (1981), no. 1, 55–74.

[28] A. an Huef, M. Laca, I. Raeburn, and A. Sims, *KMS states on the C^*-algebras of reducible graphs*, to appear in Ergodic Theory and Dynamical Systems, http://dx.doi.org/10.1017/etds.2014.52, in press, arXiv: 1402.0276 [math.OA].

[29] ———, *KMS states on the C^*-algebras of finite graphs*, Journal Mathematical Analysis and Applications **405** (2013), no. 2, 388–399.

[30] T. Kajiwara and Y. Watatani, *KMS states on finite-graph C^*-algebras*, Kyushu Journal of Mathematics **67** (2013), no. 1, 83–104.

[31] S. Kaliszewski, J. Quigg, and I. Raeburn, *Skew products and crossed products by coactions*, Journal of Operator Theory **46** (2001), no. 2, 411–433.

[32] T. Katsura, *AF-embeddability of crossed products of Cuntz algebras*, Journal of Functional Analysis **196** (2002), no. 2, 427–442.

[33] ______, *The ideal structures of crossed products of Cuntz algebras by quasi-free actions of abelian groups*, Canadian Journal of Mathematics (2003), no. 6, 1302–1338.

[34] A. Kishimoto, *Simple crossed products of C^*-algebras by locally compact abelian groups*, The Yokohama Mathematical Journal **28** (1980), no. 1, 69–85.

[35] A. Kishimoto and A. Kumjian, *Simple stably projectionless C^*-algebras arising as crossed products*, Canadian Journal of Mathematics **48** (1996), no. 5, 980–996.

[36] ______, *Crossed products of Cuntz algebras by quasi-free automorphisms*, Operator Algebras and their Applications, Fields Inst. Commun **13** (1997), 173–192.

[37] A. Kumjian and D. Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Theory and Dynamical Systems **19** (1999), no. 06, 1503–1519.

[38] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific Journal of Mathematics **184** (1998), no. 1, 161–174.

[39] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algebras*, Journal of Functional Analysis **144** (1997), no. 2, 505–541.

[40] T. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, American Mathematical Society, Providence, R.I, 1997.

[41] M.H. Mann, I. Raeburn, and C.E. Sutherland, *Representations of finite groups and Cuntz-Krieger algebras*, Bulletin of the Australian Mathematical Society **46** (1992), no. 02, 225–243.

[42] C. D. Meyer, *Matrix analysis and applied linear algebra*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2000.

[43] P.S. Muhly and M. Tomforde, *Adding tails to C^*-correspondences*, Documenta Mathematica **9** (2004), 79–106.

[44] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[45] ______, *Pullback and pushout constructions in C^*-algebra theory*, Journal of Functional Analysis **167** (1999), no. 2, 243–344.

[46] I. Raeburn, *Graph algebras*, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I, 2005.

[47] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Transactions of the American Mathematical Society **356** (2004), no. 1, 39–59.

[48] M.A. Rieffel, *Continuous fields of C^*-algebras coming from group cocycles and actions*, Mathematische Annalen **283** (1989), no. 4, 631–643.

[49] L. Robert, *Classification of inductive limits of 1-dimensional NCCW complexes*, Advances in Mathematics **231** (2012), no. 5, 2802–2836.

[50] M. Rørdam, F. Larsen, and N. Laustsen, *An Introduction to K-theory for C^*-algebras*, vol. 49, Cambridge University Press, 2000.

[51] M. Rørdam and E. Størmer, *Classification of nuclear, simple C^*-algebras*, in Classification of nuclear C*-algebras. Entropy in operator algebras, Springer, 2002, pp. 1–145.

[52] L. Santiago, *Crossed products by actions of finite groups with the Rokhlin property*, International Journal of Mathematics **26** (2015), no. 7.

[53] E. Seneta, *Non-negative matrices and Markov chains*, Springer, New York, 1981.

[54] C.-L. Shen, *Automorphisms of dimension groups and the construction of AF algebras*, Pacific Journal of Mathematics **97** (1981), no. 2, 461–469.

[55] M. Tomforde, *Stability of C^*-algebras associated to graphs*, Proceedings of the American Mathematical Society **132** (2004), no. 6, 1787–1795.

[56] ______, *Structure of graph C^*-algebras and their generalizations*, Chapter in the book "Graph Algebras: Bridging the gap between analysis and algebra", Eds. G. Aranda Pino, F. Perera Domenech, and M. Siles Molina, Servicio de Publicaciones de la Universidad de Malaga, Malaga, Spain (2006), 1–59.

[57] D.P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, no. 134, American Mathematical Soc., 2007.

[58] J. Zacharias, *Quasi-free automorphisms of Cuntz-Krieger-Pimsner algebras*, C^*-algebras (Münster, 1999), Springer, Berlin, 2000, pp. 262–272. MR 1798602 (2002a:46079)

Index

Curriculum Vitae

Christopher Chlebovec

"No problem can withstand the assault of sustained thinking." - Voltaire

Education

2010 - 2016 **PhD**, *University of New Brunswick*, Fredericton, NB.
Successfully defended on June 14, 2016

title *Quasi-free actions on graph algebras: KMS States and the structure of crossed products*

supervisors Dr. Andrew J. Dean and Dr. Dan Kucerovksy

2009-2010 **Master of Science in Mathematics**, *Lakehead University*, Thunder Bay, ON.

title *Subspaces of $\ell_2(X)$ without the approximation property*

supervisor Dr. Razvan Anisca

2004-2008 **Honours Bachelor of Science - Mathematics**, *Lakehead University*, Thunder Bay, ON.

Experience

Teaching

August 2015 - May 2016 **Lecturer**, *Lakehead University*, Thunder Bay, ON.
Courses taught: MATH-1171-FA, MATH-1171-FB, MATH-2275-WA, MATH-2311-WA, MATH-3334-WA

Winter 2015 **Sessional Lecturer**, *Lakehead University*, Thunder Bay, ON.
Course taught: MATH 2275-WA

Fall 2014 **Sessional Lecturer**, *Lakehead University*, Thunder Bay, ON.
Course taught: MATH-1151-FA

Winter 2010 **Lakehead University Math Assistant Centre Tutor**, *Lakehead University*, Thunder Bay, ON.

2009-2010 **Graduate Teaching Assistant**, *Lakehead University*, Thunder Bay, ON.

2007-2008 **Undergraduate Teaching Assistant**, *Lakehead University*, Thunder Bay, ON.

Research

June - August, 2015 **Research Associate**, *Lakehead University*, Thunder Bay, ON.
Worked with Andrew J. Dean.

Summers of 2006, 2007, 2008 **Research Assistant**, *Lakehead University*, Thunder Bay, ON.
Funded mainly by the 2006, 2007 and 2008 NSERC Undergraduate Student Research Award. I gained knowledge and experience in higher mathematics and mathematical research.

Volunteer Work

2016 **Regional Coordinator**, *Canadian Math Kangaroo Contest*, Lakehead University, Thunder Bay, ON.
Organizer of this year's Math Kangaroo Contest:
http://www.tbnewswatch.com/Artsentertainment/384962/Kids_leap_into_Kangaroo_Math_Contest_

2015 **Volunteer**, *Canadian Math Kangaroo Contest*, Lakehead University, Thunder Bay, ON.
Helped with registration and organization on the day of the contest.

2010, 2015 **Lakehead University Preview Day**, *Lakehead University*, Thunder Bay, ON.
Informed prospective students on post secondary options and provided information regarding potential careers in mathematics.

2007-2010, 2015 **TD Canada Trust Northwestern Ontario High School Mathematics Competition**, *Lakehead University*, Thunder Bay, ON.
Helped formulate questions, invigilate and mark.

Publications

2016 C. Chlebovec, *KMS states for quasi-free actions on finite-graph algebras*, Journal of Operator Theory, **75** (2016), no. 1, 119-138.

2012 R. Anisca, C. Chlebovec, *Subspaces of $\ell_2(X)$ without the approximation property*, Journal of Mathematical Analysis and its Applications **395** (2012), 523-530.

2012 R. Anisca, C. Chlebovec, M. Ilie, *On the structure of arithmetic sums of affine Cantor sets*, Real Analysis Exchange **37** (2012), 1-8.

2009 R. Anisca, C. Chlebovec, *On the structure of arithmetic sums of Cantor sets with constant ratios of dissection*, Nonlinearity **22** (2009), 2127-2140.

Conferences

June 15-19, 2015 **Canadian Operator Symposium on Operator Theory and Operator Algebras (COSy)**, *University of Waterloo*, Waterloo, ON.

July 9-12, 2008 **Canadian Undergraduate Mathematics Conference (CUMC)**, *University of Toronto*, Toronto, ON.

Presentations

Conference Talks

June 15-19, 2015 **KMS states for quasi-free actions on finite graph algebras**, *Canadian Operator Symposium on Operator Theory and Operator Algebras (COSy)*, University of Waterloo, Waterloo, ON.
20 minute talk

July, 2008 **Sums of Cantor Sets**, *Canadian Undergraduate Mathematics Conference (CUMC)*, University of Toronto, Toronto, ON.
40 minute talk

University Talks

2015 **Colloquium**, *KMS states for quasi-free actions on finite graph algebras*, Lakehead Unviersity, Department of Mathematics, Thunder Bay, ON.
40 minute talk

2011-2013 **Seminar Talks**, University of New Brunswick, Department of Mathematics, Fredericton, NB.
50 minute talks

October, 2008 **Colloquium**, *Sums of Cantor Sets*, Lakehead Unviersity, Department of Mathematics, Thunder Bay, ON.
40 minute talk

Scholarships and Awards

2010 **Governor-General's Gold Medal**

description Awarded to the graduate student who achieves the highest academic standing in a Master's degree program at Lakehead University.

2009-2010 **NSERC Post Graduate Scholarship Masters (PGS M) extension**

2008-2009 **NSERC Post Graduate Scholarship Masters (PGS M)**

2008 **C.J. Sanders Graduate Award in Mathematics**

2008 **Ontario Graduate Fellowship**

2007-2008 **Arthur and Ruth Kajander Merit Award**

description Applicants will be considered on the basis of academic excellence in Math and Science, demonstrated leadership and social responsibility and an appreciation for the impact of a well-rounded liberal education on an individual's quality of life and personal development.

2006, 2007, 2008 **NSERC Undergraduate Student Research Award**

Computer Skills

Tools LaTeX, Adobe Acrobat 9 Pro Extended, Microsoft Word, Microsoft Excel, Microsoft Powerpoint

References

Available Upon Request