THE EXACT COST OF EXPLORING STREETS WITH A CAB

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The Exact Cost of Exploring Streets with a CAB*

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Abstract

A fundamental problem in robotics is to compute a path for a robot from its current location to a given target. In this paper we consider the problem of a robot equipped with an on-board vision system searching for a target t in an unknown environment.

We assume that the robot is located at a point s in a polygon that belongs to the well investigated class of polygons called *streets*. A *street* is a simple polygon where s and t are located on the polygon boundary and the part of the polygon boundary from s to t is weakly visible to the part from t to s and vice versa.

We are interested in the ratio of the length of the path traveled by the robot to the length of the shortest path from s to t which is called the competitive ratio of the strategy. In this work we present the first *exact* analysis of the *continuous angular bisector* (*CAB*) strategy, which has been considered several times before, and show that it has a competitive ratio of ≈ 1.6837 in the worst case.

1 Introduction

Finding a path from a starting location to a target within a given scene is an important problem in robotics. A natural and realistic setting is to assume that the robot has only a partial knowledge of its surroundings and that the amount of information available to the robot increases as it travels and discovers its surroundings. For this purpose, the robot is equipped with an on-board vision system that provides the visibility map of its local environment. The robot uses this information to devise a search path for a visually identifiable target located outside the current visibility region. The quality of a search strategy is then evaluated under the framework of competitive analysis for on-line searches, as introduced by Sleator and Tarjan [27]. A search strategy is called *c-competitive* if the path traveled by the robot to find the target is at most c times longer than a shortest path. The parameter c is called the *competitive ratio* of the strategy.

As can easily be seen, there is no strategy with a competitive ratio of $o(n)$ for scenes with arbitrary obstacles having a total of n vertices [4] even if we restrict ourselves to searching in a simple polygon. Therefore, the on-line search problem has been studied previously in various contexts where the geometry of the obstacles is restricted such as searching in special classes of simple polygons [7, 8, 13, 22, 23], among rectangles [2, 3, 4, 5, 24, 25], convex polygons [14], and on the real line [1, 9, 10].

In this paper we study a competitive strategy to search in *street polygons*. In a street P the starting point s and the target t are located on the boundary of P and all points in P are visible from some point on the shortest path from s to t.

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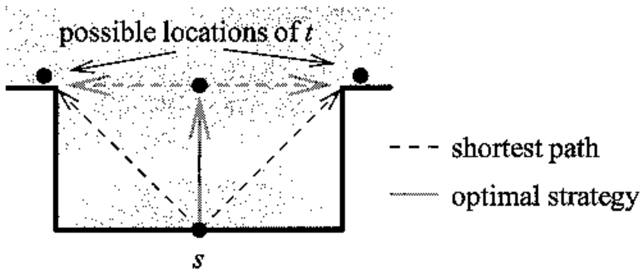

Figure 1: A lower bound for searching in rectilinear streets.

The class of street polygons was first introduced by Klein, and he was also the first to present a search strategy for streets [15]. His strategy *lad* is based on the idea of minimizing the *l*ocal *a*bsolute *d*etour. He gives an upper bound on its competitive ratio of $1+3/2\pi\,(\sim 5.71)$. The upper bound on the competitive factor was later improved by Icking to $1+\pi/2+\sqrt{1+\pi^2/4}$ (~ 4.44) [11].

A number of other strategies have been presented since by Kleinberg [16], López-Ortiz and Schuierer [19, 20, 21], Semrau [26], Dasgupta *et al.* [6], and Kranakis and Spatharis [17]. Unfortunately, the analyses of the last two results turned out to be erroneous. The currently best known competitive ratio is ≈ 1.51 [12].

Due to the simple lower bound example shown in Figure 1 there is no strategy with a competitive ratio less than $\sqrt{2}$ [15]. If a strategy moves to the left or right before seeing t, then t can be placed on the opposite side, thus forcing the robot to travel more than $\sqrt{2}$ times the diagonal. Interestingly, this example also provides a lower bound for randomized strategies. To see this consider the expected distance of a randomized strategy to the left possible location of t once the robot reaches the dashed line segment. If the expected distance is less than one we place the target on the right side, otherwise on the left. Clearly, the expected length of the path generated the randomized strategy is also at least 2.[1] Curiously enough, this is the only known lower bound even for arbitrarily oriented streets. It remains an open question whether there is an optimal competitive strategy, that is, a strategy whose competitive ratio matches the lower bound.

In this paper we present an exact analysis the strategy *continuous angular bisector* (*CAB*) which has been considered independently by several authors [6, 18, 21]. We show that the competitive ratio of *CAB* is ≈ 1.6837. The previously best known upper bound on the competitive ratio is 2.03 [21]. López-Ortiz shows a lower bound of ≈ 1.6837 on the competitive ratio of *CAB* if only triangles are considered [18]. We show that *CAB* is no worse even in general streets.

CAB is a very natural strategy where the robot walks on a curve such that, at any moment, the direction it is facing always bisects its visibility angle.[2] It is somewhat surprising that *CAB* can be analysed exactly as *CAB* consists of hyperbolic arcs whose length cannot always be expressed in a closed form.

The importance of *CAB* is threefold:

- it compares favourably to most other strategies proposed [15, 16, 19, 20, 21],
- it is a C^1-*continuous* strategy in large parts of the polygon, as opposed to all others which may contain many more bends; thus, a robot may follow a *CAB* path without having to stop as often, and

[1] The same observation also holds for biased strategies as introduced in [22].

[2] The visibility angle is defined below.

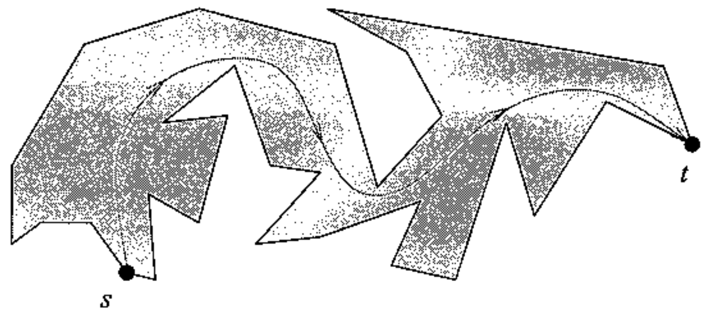

Figure 2: A street polygon.

- is used as a component of hybrid strategies for searching in streets as well as other domains, such as, for example, to search for the kernel of a polygon [13].

2 Searching for a goal in a street

We assume that the robot can be modeled by a point that is contained in a simple polygon P in the plane. Its start position is s and the position of the target is denoted by t. We assume that both of them are vertices of P. For clarity, we repeat the definition of a street.

Definition 2.1 *A simple polygon P in the plane with two distinguished vertices s and t on its boundary is called a* street *if every point in P is visible from some point on the shortest path from s to t.*

We briefly summarize the facts about searching in streets that are revelant for us (see also [15, 19, 26]).

The competitive ratio of most strategies—and, in particular, *CAB*—depends only on the competitive ratio achieved in funnels. A *funnel* is a street which consists of two reflex chains and one line segment. A *reflex chain* is a polygonal chain all of whose vertices have an interior angle larger than 180°. The point common to both reflex chains is called the *apex of the funnel* and denoted by s, see Figure 3 for an example. We denote the other end points of the two reflex chains by u_l and u_r; the target is hidden at one of u_l or at u_r. A robot searching in a funnel will know which case applies latest when it reaches the line segment connecting u_l and u_r. In the analysis of a strategy, both cases have to be considered and the maximum of them determines the competitive ratio.

Klein shows that if a strategy achieves a competitive factor of c in funnels, then it can easily be extended to a c-competitive strategy for searching in streets [15].

While a strategy proceeds, we always denote the most advanced visible point on the left chain with v_l and the most advanced visible point on the right chain with v_r. Both v_l and v_r are vertices of P. Furthermore, let d_l and d_r denote the distances from the actual position to v_l and v_r, resp., cf. Figure 3.

We define the *visibility angle* of a point p to be the angle between the line segments from p to v_l and to v_r and denote it by γ_p, see Figure 3. The visibility angle of s is called the *opening angle* of the funnel.

Suppose that the robot moves from the point p to a point q in the triangle formed by v_l, p, and v_r. Let α be the angle $\angle p v_l q$ and β be the angle $\angle q v_r p$.

Observation 2.1 *The new visibility angle γ_q at q is given by $\gamma_q = \gamma_p + \alpha + \beta$.*

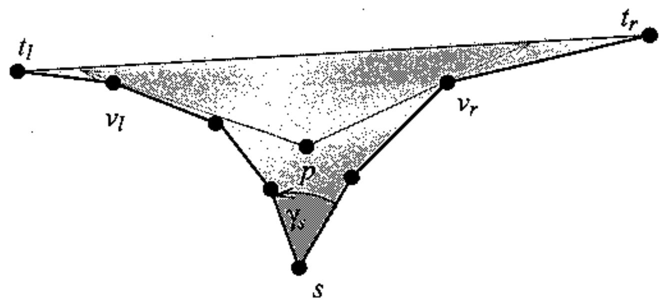

Figure 3: A funnel polygon.

3 Walking on Angular Bisectors

A very natural idea for a strategy is the following [6, 13, 18].

> **Strategy *CAB* (continuous angular bisector):**
> The robot walks along the curve C such that, for all points p on C, the direction of the motion of the robot (that is the tangent to C in p) bisects the angle $\angle v_l p v_r$.

Note that one of the points v_l and v_r changes when the path of the robot intersects a line that is collinear with one of the edges of the funnel. However, the visibility angle does not change. Hence, the curve generated by *CAB* is C^1-continuous.

The path described by Strategy *CAB* consists of hyperbolic arcs with foci a v_l and v_r. Let $d(p,q)$ is the length of the shortest path from p to q in P and $|pq|$ the length of the line segment from p to q. The points p that lie on the path generated by *CAB* satisfy the equation

$$|pv_l| - |pv_r| = d(s, v_l) - d(s, v_r)$$

where s is again the apex of the funnel. It is interesting to note that this is the same equation that is obtained for the Strategy *clad* [21]. As mentioned before, López-Ortiz and Schuierer show that the competitive ratio of *clad* (and, thus, *CAB*) is 2.03. Dasgupta *et al.* also investigate the competitive ratio of *CAB* and claim a competitive ratio of 1.7 [6]. Unfortunately, their analysis contains a flaw. In this paper we make use of their approach and correct the error in their analysis which leads to a competitive ratio of ≈ 2.38. We, furthermore, refine their analysis to achieve a competitive ratio of ≈ 1.6837 which is the exact competitive ratio of *CAB* as there is a funnel for which the competitive ratio of *CAB* achieves this value.

3.1 Analysis of the Strategy *CAB*

In this section we analyse the competitive ratio of *CAB*. Without loss of generality we assume in the following that the target t is located at u_l. We first observe that the shortest path from the current robot position p to t consists of the line segment from p to v_l and the left boundary chain from v_l to u_l. Since it can be shown that the distance from p to v_l decreases as the robot follows the curve C generated by *CAB* [6, 21], so does the distance from the robot to t as long as the v_l does not change. If v_l changes at the point p, then p can be regarded as the new apex of the funnel and the distance to the target again decreases monotonically.

In the following we analyse how much the detour increases as the robot follows C. We follow closely the approach of Dasgupta *et al.* [6]. The detour of a strategy is defined as the additional distance that the robot travels by following the path of *CAB* instead of the shortest path; in other words, if $C(p,q)$ is the length of C from p to q, then the detour $det(p,q)$ between p and q is defined as

$$det(p,q) = C(p,q) - d(p,q).$$

We are interested in the detour $det(s,t)$ between s and t. Note that competitive ratio c of *CAB* is given by

$$c = 1 + \frac{det(s,t)}{d(s,t)}.$$

Our analysis is incremental. Assume that the robot moves from a point p to a point q and that v_l does not change on the part of C from p to q. How much does the detour increase? The detour up to the point p is given by $det_p(s,t)$ from s to t via p is

$$det_p(s,t) = C(s,p) + d(p,t) - d(s,t) = C(s,p) + |pv_l| - d(s,v_l).$$

For the last equality we have made use of the observation that the shortest path from s to t and the shortest path from p to t both go through v_l. The detour increase $\Delta det(p,q)$ by going from p to q is therefore

$$\begin{aligned} \Delta det(p,q) &= det_q(s,t) - det_p(s,t) \\ &= C(s,q) + |qv_l| - d(s,v_l) - (C(s,p) + |pv_l| - d(s,v_l)) \\ &= C(p,q) + |qv_l| - |pv_l|. \end{aligned}$$

Hence, if the robot moves from p to q, then the detour increases by $C(p,q) + |qv_l| - |pv_l|$. Note that the above analysis is independent of the strategy we have chosen. In the following we consider a strategy which slightly differs from *CAB*. In the new strategy the robot moves along the angular bisector of its current visibility angle in a straight line for some (short) distance. Then, it computes the angular bisector of its new position and follows it for some distance and so on.

To be more precise, let p be a point on C at which v_l or v_r changes and q a point at which v_l or v_r changes the next time so that along the part of C from p to q the points v_l and v_r remain the same. Let α be the angle $\angle pv_lq$. We divide α into n parts and the new strategy now starts at $p_0 = p$ and moves along the bisector of the visibility angle of p_0 to a point p_1 such that the angle $\angle p_0v_lp_1$ equals $d\alpha = \alpha/n$. At p_1 it moves along the bisector of the visibility angle of p_1 to a point p_2 such that the angle $\angle p_1v_lp_2$ again equals $d\alpha$ and so on. It finally reaches the line through v_l and q in a point p_n which is different from q. It is easy to see that if n goes to infinity, then the curve C_n described by the new strategy converges to the part of C from p to q. In the following we calculate an upper bound on the increase in detour incurred by C_n. We denote the visibility angle of point p_i by γ_i and the distance between p_i and v_l by d_i. Furthermore, let $\theta_i = \gamma_i/2$.

Using the law of sines we can see that the increase in the detour for C_n if we move from p_i to p_{i+1} is (see Figure 4)

$$\begin{aligned} |p_ip_{i+1}| + d_{i+1} - d_i &= d_i\left(\frac{\sin(d\alpha) + \sin(\theta_i)}{\sin(\pi - \theta_i - d\alpha)} - 1\right) \\ &= d_i\left(\frac{\sin(d\alpha) + \sin(\theta_i) - \sin(\theta_i + d\alpha)}{\sin(\theta_i + d\alpha)}\right) \end{aligned}$$

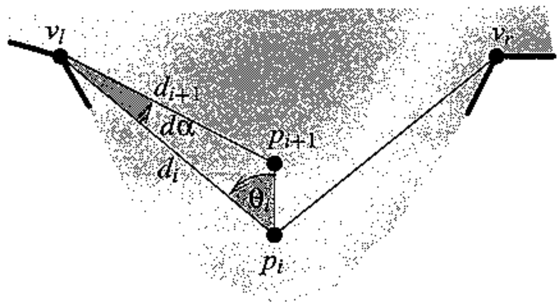

Figure 4: The computation of the increase in the detour when moving from p_i to p_{i+1}.

$$
\begin{aligned}
&= d_i \left(\frac{\sin(d\alpha) + \sin(\theta_i) - \sin(\theta_i)\cos(d\alpha) - \cos(\theta_i)\sin(d\alpha)}{\sin(\theta_i + d\alpha)} \right) \\
&= d_i \left(\frac{1-\cos(\theta_i)}{\sin(\theta_i + d\alpha)} \sin(d\alpha) + \frac{2\sin(\theta_i)}{\sin(\theta_i + d\alpha)} \sin^2(d\alpha/2) \right).
\end{aligned}
$$

Hence, the total increase I_n in detour from p_0 to p_n is given by

$$
I_n = \sum_{i=0}^{n-1} d_i \left(\frac{1-\cos(\theta_i)}{\sin(\theta_i + d\alpha)} \sin(d\alpha) + \frac{2\sin(\theta_i)}{\sin(\theta_i + d\alpha)} \sin^2(d\alpha/2) \right).
$$

Let I_{pq} denote the increase in detour of CAB if the robot goes from p to q. Since the path of the new strategy converges to the path of CAB, I_n converges to I_{pq} as n goes to infinity. Hence, the detour of the CAB strategy is given by

$$
\begin{aligned}
I_{pq} &= \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \left(\frac{1-\cos(\theta_i)}{\sin(\theta_i + d\alpha)} \sin(d\alpha) + \frac{2\sin(\theta_i)}{\sin(\theta_i + d\alpha)} \sin^2(d\alpha/2) \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \left(\frac{1-\cos(\theta_i)}{\sin(\theta_i) + O(d\alpha)} (d\alpha - O(d\alpha^3)) + \frac{2\sin(\theta_i)}{\sin(\theta_i) + O(d\alpha)} (d\alpha/2 - O(d\alpha^3))^2 \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \left(\frac{1-\cos(\theta_i)}{\sin(\theta_i) + O(d\alpha)} d\alpha \right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \left(\frac{1-\cos(\theta_i)}{\sin(\theta_i)} d\alpha \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \tan\left(\frac{\theta_i}{2}\right) d\alpha.
\end{aligned}
$$

In order to calculate the above sum we need to determine how θ_{i+1} depends on θ_i and $d\alpha$.[3] By Observation 2.1 we have that

$$
\gamma_{i+1} = \gamma_i + d\alpha + d\beta_i
$$

where $d\beta_i$ is the angle $\angle p_{i+1} v_r p_i$. Hence, $\theta_{i+1} = \theta_i + d\alpha/2 + d\beta_i/2$, for $0 \le i \le n-1$. By a simple induction we have $\theta_n = \theta_{n-1} + d\alpha/2 + d\beta_{n-1}/2 = \cdots = \theta_i + (n-i)d\alpha/2 + \sum_{j=i}^{n-1} d\beta_j/2$ and, therefore, $\theta_i = \theta_n - (n-$

[3] At this point Dasgupta *et al.* assume that $\theta_{i+1} = \theta_i + d\alpha$ [6] which is not true in general. In this way they arrive at a detour that is bounded by $d(s,t)\int_{\gamma_s}^{\pi/2} \tan(\theta/2)d\theta = d(s,t)\ln(1+\cos(\gamma_s/2))$ which is too low by a factor of two (see also Equation 3).

$i)d\alpha/2 - \sum_{j=i}^{n-1} d\beta_j/2 = \theta_n - \alpha/2 + i\,d\alpha/2 - \beta_i/2$ where $\beta_i = \sum_{j=i}^{n-1} d\beta_j$, for $0 \le i \le n-1$. Since tan increases monotonically on $[0, \pi/2]$,

$$
\begin{aligned}
I_{pq} &= \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \tan\left(\frac{\theta_n - \alpha/2}{2} + i\frac{d\alpha}{4} - \frac{\beta_i}{4}\right) d\alpha && (1)\\
&\le \lim_{n\to\infty} \sum_{i=0}^{n-1} d_i \tan\left(\frac{\theta_n - \alpha/2}{2} + i\frac{d\alpha}{4}\right) d\alpha. && (2)
\end{aligned}
$$

If we now observe that *CAB* approaches the target t monotonically, that is $d_i \le d_0$, for $1 \le i \le n-1$, then we obtain

$$
I_{pq} \le \lim_{n\to\infty} d_0 \sum_{i=0}^{n-1} \tan\left(\frac{\theta_n - \alpha/2}{2} + i\frac{d\alpha}{4}\right) d\alpha.
$$

Since $\lim_{n\to\infty} \theta_n = \gamma_q/2$, the above limit equals the integral

$$
d_0 \int_{\gamma_q-\alpha}^{\gamma_q} \tan\frac{\theta}{4}\, d\theta = d_0 \int_{\gamma_p+\beta}^{\gamma_q} \tan\frac{\theta}{4}\, d\theta \le d_0 \int_{\gamma_p}^{\gamma_q} \tan\frac{\theta}{4}\, d\theta.
$$

Recall that γ_p is the visibility angle of p and γ_q is the visibility angle of q.

If we consider the total detour, then we just need to take into account that $d_0 \le d(s,t)$. Now we can sum over the integrals between the points where v_l or v_r changes. This leads to the bound

$$
det(s,t) \le d(s,t) \int_{\gamma_s}^{\gamma_q} \tan\frac{\theta}{4}\, d\theta
$$

where γ_s is the opening angle of the funnel and γ_q is the visibility angle at which t is seen. Since $\gamma_q \le \pi$,

$$
\begin{aligned}
det(s,t) \le d(s,t) \int_{\gamma_s}^{\pi} \tan\frac{\theta}{4}\, d\theta &= d(s,t)\, 2\ln\left(1 + \cos\frac{\gamma_s}{2}\right) && (3)\\
&\le 2\ln 2\, d(s,t) \approx 1.3863\, d(s,t)
\end{aligned}
$$

Since $d(s,t)$ is the length of the shortest path from s to t, we obtain a competitive ratio of $1 + 2\ln 2 \le 2.3863$.

3.2 Improving the Analysis

The above analysis gives a very coarse bound on the competitive ratio of *CAB* which is even worse than the bound 2.03 obtained by López-Ortiz and Schuierer who use a completely different approach [21]. The slack in the above analysis is due to the fact that we have neglected the decrease in the distance d_i. The distance d_i is given by the following recurrence equation[4]

$$
d_{i+1} = \frac{\sin(\theta_i)}{\sin(\theta_i + d\alpha)} d_i = d_0 \prod_{j=0}^{i} \frac{\sin(\theta_i)}{\sin(\theta_i + d\alpha)} = d_0 \prod_{j=0}^{i} \frac{\sin\left(\theta_n - \frac{\alpha}{2} + j\frac{d\alpha}{2} - \frac{\beta_j}{2}\right)}{\sin\left(\theta_n - \frac{\alpha}{2} + j\frac{d\alpha}{2} - \frac{\beta_j}{2} + d\alpha\right)}. \quad (4)
$$

[4] A similar equation for the special case $\theta_{i+1} = \theta_i + d\alpha/2$, that is $\beta_i = 0$ is given by López-Ortiz [18].

3.2.1 An Example

For didactic purposes, let us analyze the case of an isoceles triangle. In this case, the CAB strategy produces a straight line up the bisector of the initial angle. From basic trigonometry it follows that the detour is given by $d_0(\cos(\gamma_p/2) - 1 + (1-\cos(\gamma_q/2))\sin(\gamma_p/2)/\sin(\gamma_q/2))$.

If, on the other hand, we use Equation 1 and Equation 4 we first notice that, in this case $d\alpha = d\beta_i$ for all $0 \leq i \leq n-1$. This implies that $\beta_j = \sum_{i=j}^{n-1} d\beta_i = (n-j)d\alpha = \alpha - jd\alpha$. Thus, Equation 4 becomes

$$d_{i+1} = d_0 \prod_{j=0}^{i} \frac{\sin(\theta_n - \alpha + jd\alpha)}{\sin(\theta_n - \alpha + jd\alpha + d\alpha)}.$$

Notice that this is a telescopic product and we obtain

$$d_{i+1} = d_0 \frac{\sin(\theta_n - \alpha)}{\sin(\theta_n - \alpha - (i+1)\,d\alpha)}.$$

Now, substituing in Equation 1 gives

$$\begin{aligned} I_{pq} &= \lim_{n\to\infty} \sum_{i=0}^{n-1} d_0 \frac{\sin(\theta_n - \alpha)\tan\left(\frac{\theta_n-\alpha}{2} + i\frac{d\alpha}{2}\right)}{\sin(\theta_n - \alpha + (i+1)\,d\alpha)} d\alpha \\ &= d_0 \sin(\theta_n - \alpha) \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\tan\left(\frac{\theta_n-\alpha}{2} + i\frac{d\alpha}{2}\right)}{\sin(\theta_n - \alpha + (i+1)d\alpha)} d\alpha \\ &= d_0 \sin(\theta_n - \alpha) \int_{\gamma_q/2-\alpha}^{\gamma_q/2} \frac{\tan(\theta/2)}{\sin(\theta)} d\theta \\ &= d_0 \sin(\theta_n - \alpha) \left(\frac{\sin(\gamma_q/4)}{\cos(\gamma_q/4)} - \frac{\sin(\gamma_p/4)}{\cos(\gamma_p/4)} \right). \end{aligned}$$

A sequence of trigonometric transformations shows that this value indeed *matches* the one computed previously. In particular, if we consider the "funnel" in Figure 1 where $\gamma_p = \pi/2$ and $\gamma_q = \pi$, then we obtain a detour of $\sqrt{2}-1$, as expected. This indicates that by using Equation 4 we may, in fact, obtain the *exact* detour of CAB.

3.2.2 The General Case

Now we consider the general case. Note that $\sin(\theta_i)/\sin(\theta_i + d\alpha)$ is monotonically increasing in θ_i, for $0 \leq \theta_i + d\alpha \leq \pi/2$, and, hence, we obtain from Equation 4

$$d_{i+1} \leq d_0 \prod_{j=0}^{i} \frac{\sin\left(\theta_n - \frac{\alpha}{2} + j\frac{d\alpha}{2}\right)}{\sin\left(\theta_n - \frac{\alpha}{2} + j\frac{d\alpha}{2} + d\alpha\right)}.$$

Once again, this is a telescopic product, in which the denominator of the jth term cancels with numerator of the $(j+2)$nd term of the above product and we obtain

$$d_{i+1} \leq d_0 \frac{\sin\left(\theta_n - \frac{\alpha}{2}\right)\sin\left(\theta_n - \frac{\alpha}{2} + \frac{d\alpha}{2}\right)}{\sin\left(\theta_n - \frac{\alpha}{2} + (i-1)\frac{d\alpha}{2} + d\alpha\right)\sin\left(\theta_n - \frac{\alpha}{2} + i\frac{d\alpha}{2} + d\alpha\right)}. \quad (5)$$

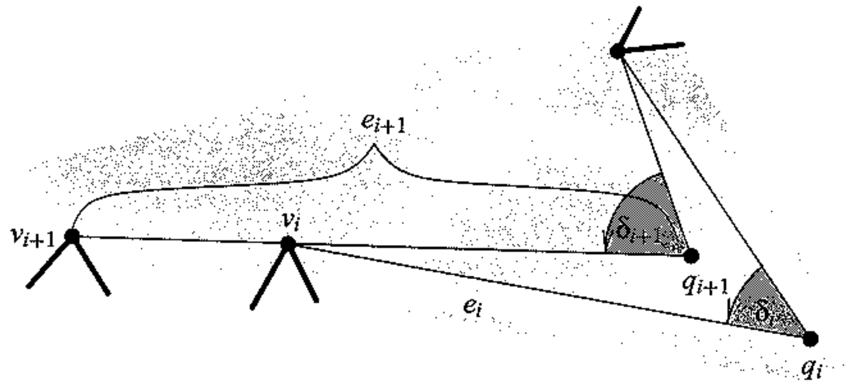

Figure 5: The vertex v_l changes.

We can now plug in the upper bound for d_i into Equation 2, which gives

$$
\begin{aligned}
I_{pq} &\leq \lim_{n\to\infty}\sum_{i=0}^{n-1} d_0 \frac{\sin\left(\theta_n - \frac{\alpha}{2}\right)\sin\left(\theta_n - \frac{\alpha}{2} + \frac{d\alpha}{2}\right)\tan\left(\frac{\theta_n - \alpha/2}{2} + i\frac{d\alpha}{4}\right)}{\sin\left(\theta_n - \frac{\alpha}{2} + (i-2)\frac{d\alpha}{2} + d\alpha\right)\sin\left(\theta_n - \frac{\alpha}{2} + (i-1)\frac{d\alpha}{2} + d\alpha\right)} d\alpha \\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} d_0 \frac{\sin^2\left(\theta_n - \frac{\alpha}{2}\right)\tan\left(\frac{\theta_n-\alpha/2}{2} + i\frac{d\alpha}{4}\right)}{\sin^2\left(\theta_n - \frac{\alpha}{2} + (i-2)\frac{d\alpha}{2} + d\alpha\right)} d\alpha.
\end{aligned}
$$

Since again $\lim_{n\to\infty}\theta_n = \gamma_q/2$, the above limit equals the integral

$$
d_0 \sin^2\left(\frac{\gamma_q - \alpha}{2}\right)\int_{\gamma_q-\alpha}^{\gamma_q}\frac{\tan(\theta/4)}{\sin^2(\theta/2)}d\theta. \tag{6}
$$

In order to obtain an upper bound on the total detour of C in P we consider all the points q_i, $0 \leq i \leq k$, at which the most advanced visible point on the left or right chain changes. We start with $q_0 = s$ and end with the point q_k at which the target becomes visible. Let δ_i be the visibility angle of q_i and v_i the most advanced visible point for q_i on the left chain. Moreover, let e_i be the distance from q_i to v_i and α_i the angle at v_i between q_i and q_{i+1}. By Expression 6 the total detour is bounded by the sum

$$
\sum_{i=0}^{k-1} e_i \sin^2\left(\frac{\delta_{i+1} - \alpha_i}{2}\right)\int_{\delta_{i+1}-\alpha_i}^{\delta_{i+1}}\frac{\tan(\theta/4)}{\sin^2(\theta/2)}d\theta. \tag{7}
$$

By Inequality 5 the distance e_i, for $1 \leq i \leq k-1$, is bounded by (see Figure 5)

$$
e_i \leq e_{i-1}\frac{\sin^2((\delta_i - \alpha_{i-1})/2)}{\sin^2(\delta_i/2)} + |v_i v_{i+1}|,
$$

with $e_0 = |v_0 v_1|$. If we solve the above recurrence, we obtain that

$$
e_i \leq \sum_{l=0}^{i}\prod_{j=l+1}^{i}\frac{\sin^2((\delta_j - \alpha_{j-1})/2)}{\sin^2(\delta_j/2)}|v_l v_{l+1}|,
$$

for $0 \le i \le k-1$. If we plug the expression for e_i into (7) and extract all the terms that are multiplied by $|v_l v_{l+1}|$, then we obtain the sum

$$S_l = |v_l v_{l+1}| \sum_{i=l}^{k-1} \prod_{j=l+1}^{i} \frac{\sin^2((\delta_j - \alpha_{j-1})/2)}{\sin^2(\delta_j/2)} \sin^2\left(\frac{\delta_{i+1}-\alpha_i}{2}\right) \int_{\delta_{i+1}-\alpha_i}^{\delta_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta.$$

We see as before that $\delta_i \le \delta_k - \sum_{j=i}^{k-1} \alpha_j = \phi_i$. Since $\phi_i \le \pi$, and the function $\sin((\delta_j - \alpha_{j-1})/2)/\sin(\delta_j/2)$ is monotone in δ_j, if $0 \le \delta_j \le \pi$, S_l is bounded by

$$S_l \le |v_l v_{l+1}| \sum_{i=l}^{k-1} \prod_{j=l+1}^{i} \frac{\sin^2((\phi_j - \alpha_{j-1})/2)}{\sin^2(\phi_j/2)} \sin^2\left(\frac{\delta_{i+1}-\alpha_i}{2}\right) \int_{\delta_{i+1}-\alpha_i}^{\delta_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta. \tag{8}$$

Note that the function

$$\frac{\sin^2((\delta_{i+1}-\alpha_i)/2)\tan((\delta_{i+1}-\alpha_i+\theta)/4)}{\sin^2((\delta_{i+1}-\alpha_i+\theta)/2)}$$

is also increasing in δ_{i+1} and, therefore,

$$\begin{aligned}
\sin^2\left(\frac{\delta_{i+1}-\alpha_i}{2}\right) \int_{\delta_{i+1}-\alpha_i}^{\delta_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta &\le \sin^2\left(\frac{\phi_{i+1}-\alpha_i}{2}\right) \int_{\phi_{i+1}-\alpha_i}^{\phi_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \\
&= \sin^2\frac{\phi_i}{2} \int_{\phi_i}^{\phi_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta
\end{aligned} \tag{9}$$

since $\phi_{i+1} - \alpha_i = \phi_i$. Using this equality and Inequality (9) in (8) we obtain

$$\begin{aligned}
S_l &\le |v_l v_{l+1}| \sum_{i=l}^{k-1} \prod_{j=l+1}^{i} \frac{\sin^2(\phi_{j-1}/2)}{\sin^2(\phi_j/2)} \sin^2\frac{\phi_i}{2} \int_{\phi_i}^{\phi_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \\
&= |v_l v_{l+1}| \sum_{i=l}^{k-1} \frac{\sin^2(\phi_l/2)}{\sin^2(\phi_i/2)} \sin^2\frac{\phi_i}{2} \int_{\phi_i}^{\phi_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \\
&= |v_l v_{l+1}| \sin^2\frac{\phi_l}{2} \sum_{i=l}^{k-1} \int_{\phi_i}^{\phi_{i+1}} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \\
&= |v_l v_{l+1}| \sin^2\frac{\phi_l}{2} \int_{\phi_l}^{\phi_k} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \\
&\le |v_l v_{l+1}| \sin^2\frac{\phi_l}{2} \int_{\phi_l}^{\pi} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta.
\end{aligned}$$

The total detour is now bounded by

$$\begin{aligned}
\sum_{l=0}^{k-1} S_l &\le \sum_{l=0}^{k-1} |v_l v_{l+1}| \sin^2\frac{\phi_l}{2} \int_{\phi_l}^{\pi} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \\
&\le \max_{0\le\phi\le\pi} \left(\sin^2\frac{\phi}{2} \int_{\phi}^{\pi} \frac{\tan(\theta/4)}{\sin^2(\theta/2)} d\theta \right) \sum_{l=0}^{k-1} |v_l v_{l+1}| \\
&= \max_{0\le\phi\le\pi} \left(\sin^2\frac{\phi}{2}\left(1 - \ln\tan\frac{\phi}{4}\right) - 1 + \cos\frac{\phi}{2} \right) d(s,t).
\end{aligned} \tag{10}$$

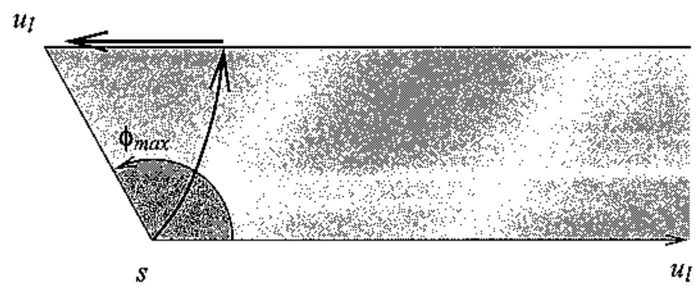

Figure 6: A triangle for which the lower bound for *CAB* is achieved.

From the derivative it can be seen that the above function has a unique maximum in the interval $[0, \pi]$ at $\phi_{max} \approx 1.86973$ and with a value of less than 0.68372. Finally, we note that all of the above inequalities change into equalities if the consider a triangle with opening angle ϕ_{max}, the distance $|su_l| = 1$, and u_r placed infinitely far away (see Figure 6). The lower bound is also shown in [18]. Hence, we have proven the following theorem.

Theorem 3.1 *The competitive ratio of* CAB *is* ≈ 1.6837.

The expression in Equation 10 gives an upper bound on the detour in terms of the starting aperture angle of a funnel. The plot in Figure 7 shows that CAB has worst case performance for initial apperture angles slightly larger than $\pi/2$.

4 Conclusions

We have analysed the strategy *CAB* to search in streets. The strategy *CAB* follows a path such that the direction of movement always bisects the current visibility angle. The resulting path is a concatenation of hyperbolic arcs. Although the length of a hyperbolic arc cannot be expressed in a closed form, the maximal competitive ratio of *CAB* can be analysed exactly—as opposed to many other strategies. We show that the competitive ratio of *CAB* is ≈ 1.6837 and we also show that there is a polygon for which this competitive ratio is assumed. *CAB* is a very natural strategy that is also considered in several other contexts. In addition, *CAB* has the advantage of being C^1-continuous in a funnel.

One of the most interesting and tantalizing open problems in the area of on-line geometric searching remains the question whether it is possible to design an optimal competitive strategy to search in streets.

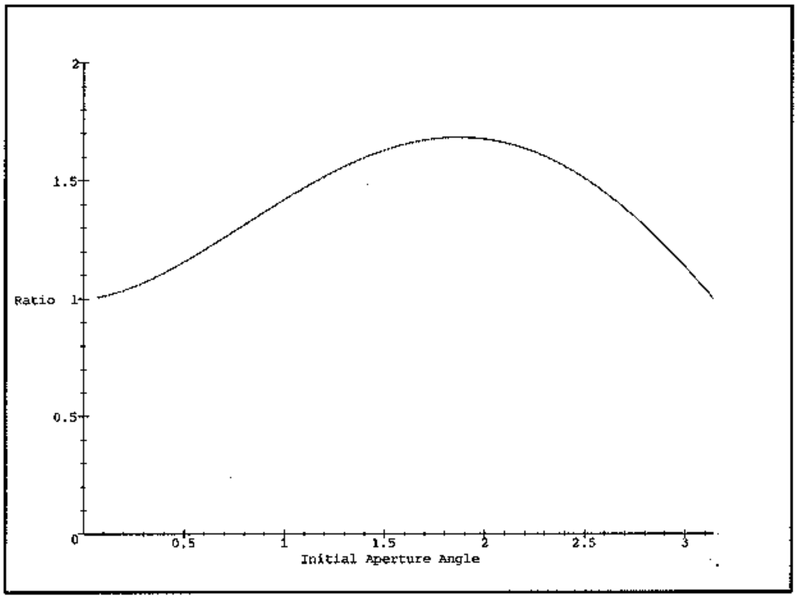

Figure 7: Upper bound on the competitive ratio of CAB.

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