FACTORING THE PRODUCT OF A CUBIC GRAPH AND A TRIANGLE

by

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Abstract

Kotzig [J. Graph Theory 3 (1979) pp 23-34] proved that for any cubic graph G and any circuit of length n, C_n, n>3, the (Cartesian) product $G \times C_n$ has a 1-factorization, and that if G contains a bridge, $G \times C_3$ does not. In this paper it is shown that if G is a 2-connected cubic graph, then $G \times C_3$ decomposes into two hamilton circuits and a 1-factor.

1. Introduction

Kotzig [Kotzig, 1979] proved that for any cubic graph G and any circuit of length n, C_n, n>3, the (Cartesian) product GxC_n has a one-factorization. He also noted that GxC_3 does not have a one-factorization if G is a cubic graph with a bridge. However the case GxC_3 when G is a two-connected cubic graph was left open. In this note, I prove that GxC_3 has a one-factorization if G is a two-connected cubic graph. In fact GxC_3 decomposes into two hamilton circuits and a one-factor.

In this paper all graphs are finite, and all original graphs have no loops or multiple edges. However in some of the analysis, graphs with loops and/or multiple edges may be created. Nevertheless we still use the notation {v,u} to mean an edge joining vertex v to vertex u in places where no confusion arises. We also identify subgraphs with their set of edges.

Define the Cartesian product of GxH of two graphs G=(V(G), E(G)) and H=(V(H), E(H)) in the following way. V(GxH) = V(G)xV(H) = {(v,x)|vεV(G) and xεV(H}, the ordinary set-theoretic Cartesian product of the vertex sets. The edge set of GxH consists of two types of edges. Let

$$E_{ver} = \{\{(v,x),\ (u,x)\}|(v,u\}\varepsilon E(G) \text{ and } x\varepsilon V(H)\},$$

and $$E_{hor} = \{\{(v,x),\ (v,y)\}|v\varepsilon V(G) \text{ and } (x,y)\varepsilon E(H)\}.$$

We call the set E_{hor} the <u>horizontal</u> edges, and E_{ver} the <u>vertical</u> edges. We define $E(GxH)=E_{hor} \cup E_{ver}$. The horizontal edges of GxH form |V(H)| isomorphic copies of H, and the vertical edges of GxH form |V(G)| copies of H. (The terms horizontal and vertical are taken from [Mahmoodian, 1981].)

Let A and B be a partition of the vertex set V(G). Then $\delta(A,B)$ is the set of edges with one end in A and the other end in B. Such a set is called a <u>cutset</u> of edges, or simply a <u>cut</u>. If $\delta(A,B)$ contains k edges, it is called a <u>k-edge cut</u>. E(A) is the set of edges with both ends in A, and (A,E(A)) is the subgraph induced by A. If A is a subset of vertices, we use G/A as a short form for G/E(A).

Let F be a subset of the set of edges of the graph G. Then the notation G/F denotes the graph obtained from G by contracting the edges in F. This is the situation in which multiple edges or loops may arise. However contracting an edge never decreases the edge connectivity of a graph, a fact that is used later in this paper.

2. Preliminary results

It is well known that if G is a two-connected cubic graph, then the edge set partitions into a one-factor M and a two-factor F [Petersen, 1891]. This result was extended to require M to include a given edge [Schonberger, 1934]. The two-factor F consists of a number of disjoint cycles, say F_1, $F_2, \ldots,$ F_k. $G x C_3$ consists of $F x C_3$ and three copies of M. The three copies of M form a one-factor. $F x C_3$ consists of $F_1 x C_3$, $F_2 x C_3, \ldots, F_k x C_3$. Next we study the structure of $F_i x C_3$ (remembering that F_i is some cycle C_k).

<u>Lemma 1</u>. For any integer k, $k \geq 5$, $C_k x C_3$ can be decomposed into two hamilton circuits. Moreover, for any set of three vertical edges {e,f,g} such that each of these edges is in a separate vertical triangle, e can be placed in one hamilton circuit while f and g are in the other hamilton circuit.

Proof. This proof requires a large number of cases, depending on whether k is even or odd, as well as the relative positions of e, f, and g. For the case k odd, the hamilton decomposition shown in Figure 1, up to translations and reflections, can be used.

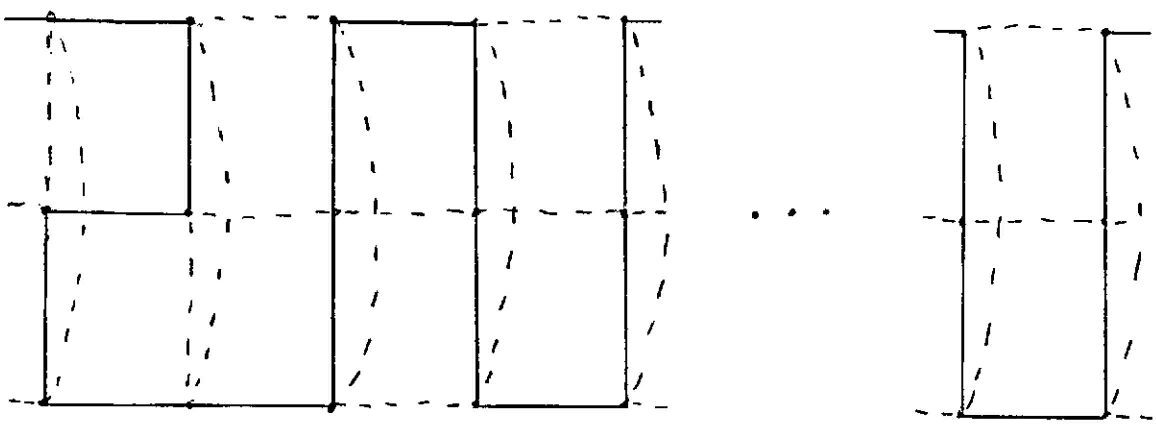

Figure 1. Hamilton decomposition of $C_k x C_3$, k odd.

Instead of talking about translation and reflections of the hamilton decomposition, we can talk about moving the edges e, f, and g as long as the relative positions of these edges is fixed.

The vertical edges can be identified by the level and horizontal location. Thus edge {(1,1), (2,1)} is in location (1,1). In general edge {(1,x), (2,x)} is in location (1,x), {(2,y, (3,y)} is at (2,y), and {(1,z), (3,z)} at (3,z). Note that the solid hamilton circuit uses the vertical edges at locations (1,1), (2,2), (1,3), (2,3),...,(1,k), (2,k). The other vertical edges are in the other hamilton circuit.

There are essentially 4 subcases to consider, depending on what relative levels e,f, and g are on.

(1) e,f,g are all on the same level. Place e in location (2,1). Then f and g are in locations (2,i) and (2,j) for $i,j \neq 1$, so are in the other circuit.

(2) e on one level, f and g on a second level. Place e in location (2,2), and f,g in (3,i), (3,j).

(3) e and f on one level, g on another. Place f in (2,1), e in (2,i) and g in (3,j).

(4) e,f, and g are all on different levels. Place e in (3,1), f and g in (2,i) and (1,j), with $i<j$.

The even case is slightly more complicated. The hamilton decomposition used is shown in Figure 2.

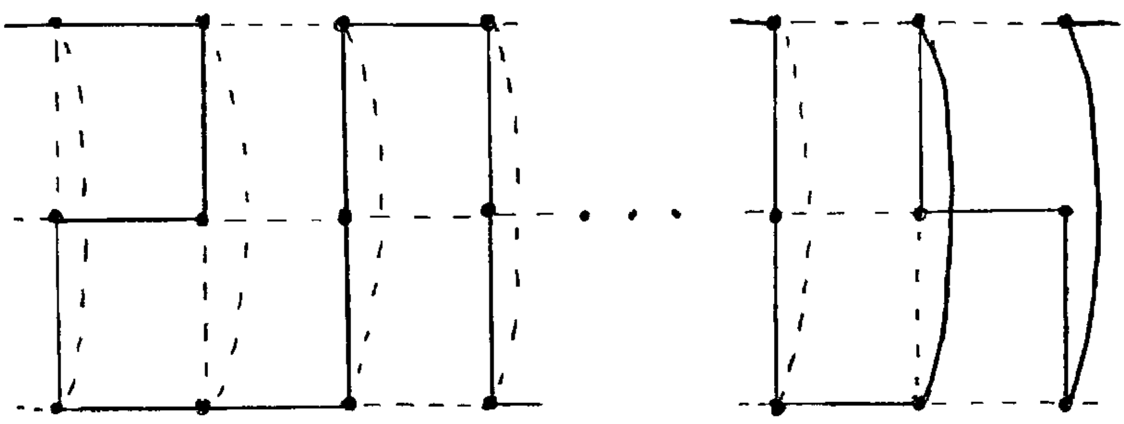

Figure 2. Hamilton decomposition of $C_k x C_3$, k even

The vertical edges used in the solid hamilton circuit are at locations (1,1), (2,1), (1,3), (2,3),..., (1,k-2), (2,k-2), (2,k-1), (3,k-1), (1,k), (3,k). The same 4 subcases must be considered as in the odd case.

(1) e,f,g are all on the same level. If e is between f and g in the circuit, put e at location (1,2) f at location (1,1), and g at (1,3). Otherwise put e at (2,1), and f and g at (2,i),(2,j) such that neither i nor j is k.

(2) e on one level, f and g on a second level. If e is between f and g, put e at (3,3) and f,g at (2,2) and (2,4). Otherwise put e at (3,1) and f,g at (2,i),(2,j), where neither i nor j is k.

(3) e and f are on one level, g on another. Place e at (2,1), f at (2,i). If i=k, then reverse so that f at (2,2). Then g is at (1,j), or (3,k-1) if j=k-1. The one problem that can happen is that j=2. But then $i \neq 2$ or k, so again the circuit can be reversed so that f is at (2,k-i+1), and g is at (1,k).

(4) e,f, and g are all on different levels. Put e at (3,1). If f or g is adjacent to e, put at (1,k) and (2,i). Otherwise put f and g at (2,i), i largest possible, and (1,j). qed.

We are interested in these 2-factorizations partly because each hamilton circuit can be split into two 1-factors. Thus in particular Lemma 1 shows that $C_k x C_3$ has a 1-factorization.

3. Related Questions

Before going on to prove the main theorem, I would like to look at the problem a different way, and its relationship to some other problems. Several problems, including this one, can be restated as cubic graph labelling problems. Perhaps the best-known such problem is whether a cubic graph G has a 1-factorization or not. The restatement is:

<u>Problem 0</u>. Can the edges of G be labelled by the set {1,2,3} such that each symbol occurs exactly one at each vertex?

Of course in general the answer is no, the Petersen graph being the smallest counterexample. The problem under discussion here is equivalent to:

<u>Problem 1</u>. Label each edge of G with ordered triples with entries from {1,2,3,4,5} such that at each vertex:

(1) each symbol occurs once or thrice;

(2) if a symbol occurs three times at a vertex, it must occur once in each coordinate of the triples.

Of course the symbols represent 1-factors, the i^{th} coordinate in the triple determining the one-factor that the corresponding horizontal edge on the i^{th} level is in. This edge-coloring can be extended to the vertical triangles if and only if the conditions of P_1 are fulfilled.

As noted before, P_1 does not have a solution if G has a bridge, a 1-edge cut, as noted in [Kotzig, 1979].

The next problem is a strengthening of Seymour's double cycle cover conjecture: any 2-connected graph contains a multi-set of circuits such that each edge occurs in exactly two of the circuits. This is equivalent to the same problem restricted to 2-connected cubic graphs. If the multiset of circuits is restricted to consisting of five families of circuits, the circuits in a family being pairwise disjoint, then this is equivalent to labelling each edge of G with a 2-subset of {1,2,3,4,5} such that each symbol occurs twice or not at all at each vertex of G. Taking complements we get:

Problem 2. Label each edge of G with 3-subsets of {1,2,3,4,5} such that each symbol occurs once or thrice at each node.

Both Problem 1 and Problem 2 imply solutions to the following:

Problem 3. Label each edge of G with a multiset of size 3 from {1,2,3,4,5} such that each symbol occurs once or thrice at each vertex.

Clearly a solution to Problem 1 leads to a solution to Problem 3 by simply replacing each ordered triple by the equivalent multiset, and a solution to Problem 2 is itself a solution to Problem 3. The converse of these statements may not be true. I know of no way to get from a labelling with multisets of size 3 to a labelling with ordinary sets of size 3. Similarly the multisets of size 3 used in a solution of Problem 2 sometimes cannot be ordered to form a solution of Problem 1. For example, the Petersen graph can be double covered by five circuits as in Figure 3. However, the resulting solution to Problem 3 cannot be made into a solution of problem 1 by ordering the triples on the edges.

A solution to Problem 0 leads easily to solutions of all three of these problems. Replace label 1 by (1,4,5), label 2 by (5,2,4), and label 3 by (4,5,3). This labelling solves Problems 1, 2 and 3 simultaneously. I do not know any 2-connected cubic graph which has no 1-factorization for which solutions to Problems 1 and 2 both lead to the same solution to Problem 3.

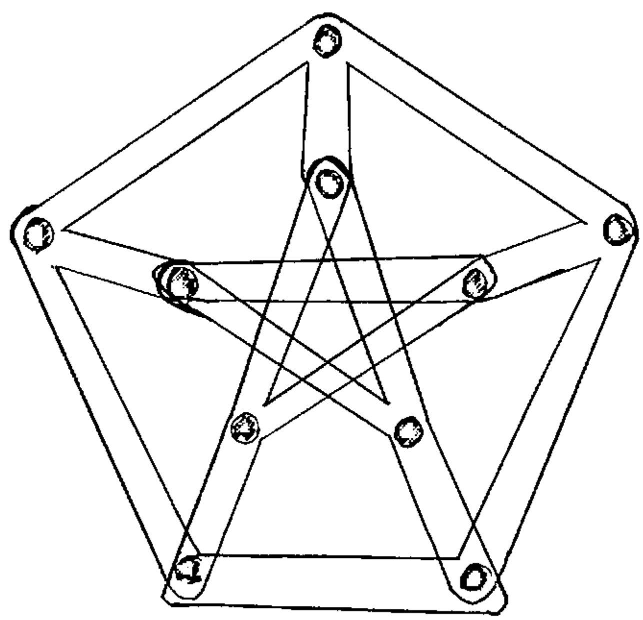

Figure 3. A Double Cycle Cover of the Petersen Graph

4. The Main Result

The approach taken in this section is quite different to that taken in the previous section. It is now possible to construct a 1-factorization of GxC_3. The 2-connected cubic graph G is factored into a 1-factor M and a 2-factor F. Then FxC_3 is factored into two 2-factors by lemma 1. The three copies of M form a 1-factor of GxC_3. Thus GxC_3 has been factored into two 2-factors and a 1-factor. Next some edges are interchanged between the 1-factor and the two-factors so that the 2-factors become hamilton circuits, and the 1-factor remains a 1-factor. But first we need a special decomposition of G.

Theorem 1. Let G be a 2-connected cubic graph with a specified edge e. There is a decomposition of G into a 1-factor M and a 2-factor F such that e is in M, and G/F contains two disjoint spanning trees. Moreover the only 4-gons in F are adjacent to two edges that form a 2-edge cut, unless G is K_4.

Proof. If the condition about the 4-gon is removed, then this theorem has already been proved [Fan, to appear]. The proof here is different, slightly more elementary, and proves a somewhat stronger result. Note that if the 4-gon condition was removed, the theorem would be false. Two copies of K_4, each with an edge removed, and replaced by two edges joining the K_4's, would be the smallest counterexample.

If one of the connecting edges were e, then F would consist of two 4-gons. However the theorem is clearly true for graphs G with 6 or fewer vertices.

As a base case, assume G is cyclically 4-edge-connected, and has no 4-gons. Then G has no 2-edge cut, no non-trivial 3-edge cut, and no 4-gons. G can be decomposed into a 1-factor M and a 2-factor F such that a given edge e is in M [Schonberger, 1934]. Now consider G/F. The circuits of F become the vertices of G/F, the edges of M are the edges of G/F. G/F has no vertex of degree less than 4, and is 4-edge-connected. It can have no vertex of degree less than 4, and is 4-edge-connected. Consider any partition of V(G/F) into parts $V_1, V_2, \ldots, V_k$. Each part is attached to all the others by at least 4 edges. Hence there must be at least 2k edges joining vertices in different components. By a theorem in [Nash-Williams, 1961], and [Tutte, 1961], G/F must contain two disjoint spanning trees.

Thus we may assume G has a 4-gon, a 2-edge cut, or a 3-edge cut. We proceed by induction on the size of G. We assume that the theorem is true for all 2-connected cubic graphs smaller than G. There are five cases.

(1) e is in a 2-edge cut $\delta(A,B)=\{e,f\}$, where $e=\{a,b\}$, $f=\{c,d\}$, a and c are in A, and b and d are in B. Define $G_A=(A,E(A)\cup\{a,c\})$ and $G_B=(B,E(B)\cup\{b,d\})$. Both G_A and G_B are 2-edge-connected cubic graphs.

By the induction hypothesis G_A has a decomposition into a 1-factor M_A and a 2-factor F_A with {a,c} in M_A, such that G_A/F_A has two disjoint spanning trees T_1 and T_2. Similarly G_B has a decomposition into a 1-factor M_B and a 2-factor F_B, with {b,d} in M_B, such that G_B/F_B has two disjoint spanning trees T_3 and T_4. As well, F_A and F_B can be assumed not to have a 4-gon adjacent to a 2-edge cut unless $G_A=K_4$ or $G_B=K_4$.Let $F=F_A\ F_B$, and let $M=M_A\ M_B$ {e,f} {{a,c},{b,d}}. Then M and F are a decomposition of G into a 1-factor and 2-factor with $e\varepsilon M$. Consider G/F. $T_1\ T_3$ {e} and $T_2\ T_4$ {f} are two disjoint spanning trees of this graph, as required. We must check all 4-gons in F. But all 4-gons of F_A or F_B are adjacent to a 2-edge cut, unless G_A or $G_B=K_4$. In this latter case, any 4-gon is adjacent to the 2-edge cut {e,f}.

(2) e is not a member of a 2-edge cut, but G has a 2-edge cut $\{f,g\}=\delta(A,B)$, with $e\varepsilon E(A)$. Let f={a,b}, g={c,d} with a, c in A and b,d in B. By the induction hypothesis, there is a decomposition of G_A, defined as in case (1), into a 1-factor M_A and a 2-factor F_A such that $e\varepsilon M_A$, and G_A/F_A has two spanning trees T_1 and T_2. As before, F_A and F_B can be assumed not to have a 4-gon adjacent to a 2-edge cut unless $G_A=K_4$ or $G_B=K_4$. There are two subcases.

(a) If the constructed edge {a,c} is in M_A, proceed precisely as in case (1).

(b) Otherwise {a,c} is in F_A. Consider G_B, defined as in case (1). Find its decomposition M_B and F_B with constructed edge {c,d} in F_B (ie, choose some other edge at c and make sure it is in M_B), and with two disjoint spanning trees T_3 and T_4 in G_B/F_B. Now $M=M_A \cup M_B$ and $F=F_A \cup F_B \cup \{f,g\} \setminus \{\{a,b\},\{c,d\}\}$ is a decomposition of G satisfying the requirements of the theorem. $T_1 \cup T_3$ and $T_2 \cup T_4$ are disjoint spanning trees of G/F.

(3) G has no 2-edge cut, but e is in a nontrivial 3-edge cut, $\{e,f,g\}=\delta(A,B)$. Consider G/B and G/A. By the induction hypothesis, G/B has a decomposition M_A, F_A such that $e \in M_A$, and $(G/B)/F_A$ has two disjoint spanning trees T_1 and T_2. Similarly G/A factors into M_B and F_B such that $e \in M_B$, and $(G/A)/F_B$ has two disjoint spanning trees T_3 and T_4. Edges f and g are in same circuit in both F_A and F_B. Letting $M=M_A \cup M_B$ and $F=F_A \cup F_B$, we get a decomposition of G into a 1-factor and a 2-factor, with e in M. One can assume e is not in T_2, or in T_4. Then $T_1 \cup T_4$ and $T_2 \cup T_3$ are spanning trees of G/F. Again no new 4-gons can be created that could contradict the theorem.

(4) G has no non-trivial 2-edge cut, but has a non-trivial 3-edge cut $\{f,g,h\}=\delta(A,B)$, and e is not in this cut. We may assume e is in E(A). Consider G/B. It has a decomposition into M_A and F_A with e in M_A, and $(G/B)/F_A$ has two spanning trees T_1 and T_2. Without loss of generality, suppose h is in M_A, and that h is not in T_2. The edge h may or may not be in T_1, but that does not matter. Next consider G/A. Find a decomposition M_B and F_B such that h is in M_B, and $(G/A)/F_B$ has two disjoint spanning trees T_3 and T_4. We may assume h is not in T_4. Now proceed as in case (3).

(5) G is cyclically 4-edge connected, but contains a 4-gon. If e is in the 4-gon, then choose some other 4-gon. If there is no other 4-gon, then F cannot contain any 4-gon, since e is not in F, and as in the base case, we are done. Let the 4-gon be (w,x,y,z), with adjacent edges {w,a}, {x,b}, {y,c}, {z,d}. See figure 4.

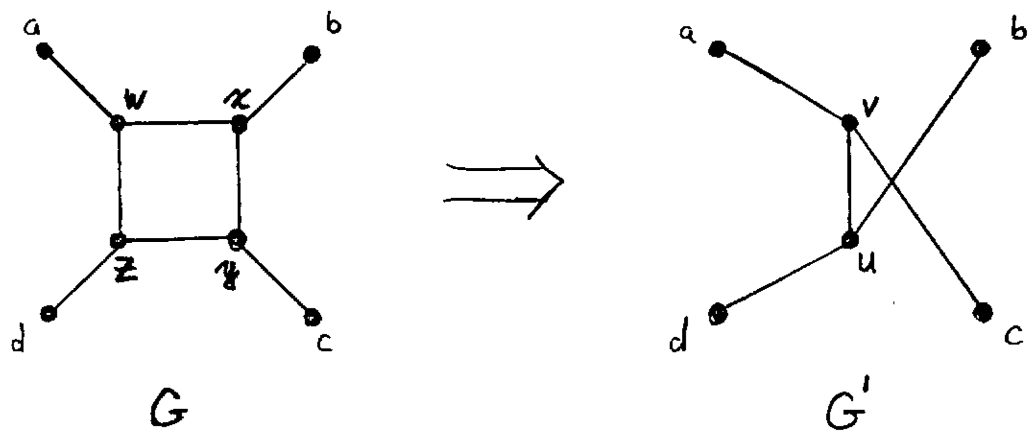

Figure 4. The reduction from G to G'

Remove these 4 vertices and 8 edges, and replace with 2 new vertices v, u and 5 edges {v,a}, {v,c}, {u,b}, {u,d}, {u,v}. Call this new graph G'. G' must factor into 1-factor M' and 2-factor F', which satisfy the conditions of the theorem. Any possible F' can be expanded into a 2-factor F of G which also satisfies the conditions of the theorem, as shown in Figure 5.

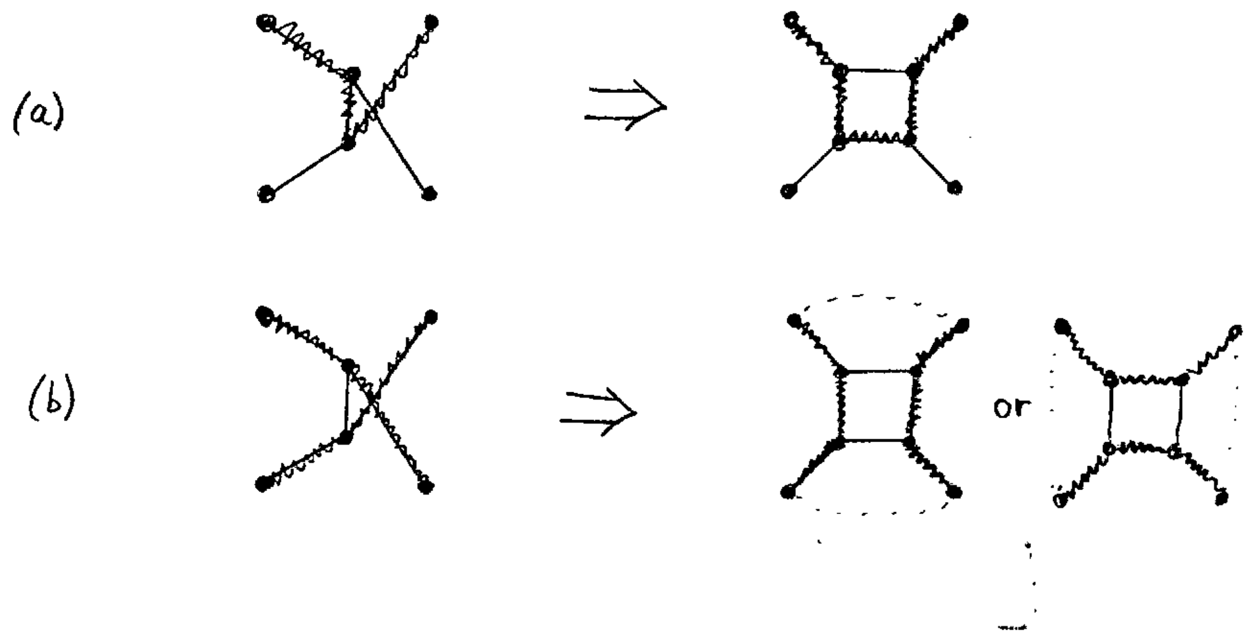

Figure 5. Expanding F' into F.

If {v,u} is in F', then G'/F' is isomorphic to G/F and hence contains two disjoint spanning trees. If {u,v} is not in F', then F can be made so that one circuit passes through all the vertices w,x,y,z. In this case G/F is isomorphic to G'/F'/{u,v}. A spanning tree in G'/F' includes a spanning tree in G/F, and so G/F has two disjoint spanning trees. No 4-gon exists in F that does not exist in F', so any 4-gon must be adjacent to a 2-edge cut. Thus F satisfies all the conditions of the theorem. qed.

Before continuing on to the main theorem, we need one more result.

<u>Lemma 2</u>. If a graph G, possibly with multiple edges, is decomposable into two disjoint spanning trees R and B, then the vertices of G can be ordered $v_1, v_2, \ldots, v_n$ such that each v_i is adjacent to at most three edges joining v_i to earlier vertices in the sequence, and at most two of these edges come from the same tree. As well, the first vertex can be chosen arbitrarily.

Proof. By induction on the order of the graph, n. Extend the induction hypothesis to include a decomposition into R and B where R and B have no circuits. It is undoubtedly true for n=1. Now consider G=(V,R B) with a given vertex v. If R and/or B is not connected, arbitrarily add edges to R(or B) until both R and B are spanning trees. R and B now contain n-1 edges each, so the average degree of a vertex is now 2(2n-2)/n which is less than 4. All vertices also have a degree of at least 2. Hence there must be at least some vertices of degree less than 4. Let v_n, $v \neq v_n$, be a vertex of degree 3 or less. This vertex is incident with at least one edge of each spanning tree, and hence at most two of either. Consider $G-v_n$, the graph with v_n and all incident edges deleted.

The remaining edges of R and B are now circuitless subgraphs of $G-v_n$. By the induction hypothesis, there is an ordering $v=v_1, v_2, \ldots, v_{n-1}$ that satisfies the condition of the theorem for $G-v_n$. Thus $v_1, v_2, \ldots, v_n$ also satisfies the condition of the theorem for G. qed.

We can now complete the construction described at the start of this section.

Theorem 2. If G is a 2-connected cubic graph, then GxC_3 can be decomposed into 2 hamilton circuits and a 1-factor.

Proof. By theorem 1, G decomposes into a 1-factor M and a 2-factor F, such that G/F contains two disjoint spanning trees R and B. Moreover, F contains no 4-gons except possibly 4-gons adjacent to a 2-edge cut. We can assume G is not K_4, since K_4xC_3 can easily be decomposed into two hamilton circuits and a 1-factor.

By lemma 2, the vertices of G/F can be ordered so that each vertex of G/F is adjacent to at most three edges of R and B, at most two from either R or B, which join them to earlier vertices in the sequence. These vertices correspond to circuits in F, and hence this orders the circuits.

Next we color the edges of GxC_3 red, blue, and white. The red edges form one hamilton circuit, the blue another, and the white form a one-factor. We start with all horizontal edges corresponding to M colored white. These do form a one-factor, but we will exchange some of the edges as we color the remaining edges, (unless F consists of a single circuit). We now color the edges of each C_ixC_3, in the order that the C_i occur in the sequence. Each C_ixC_3 is colored as described in lemma 1, so that one hamilton circuit is colored red, and the other blue. The exact orientation (translation, reflection) of this coloring depends on R,B, and how the previous C_jxC_3's are colored. If C_i and C_j are joined by an edge of R, C_j preceding C_i, then C_ixC_3 must be colored so that a red edge of C_ixC_3 "faces" a red edge of C_jxC_3 along the edge of R. By "faces" I mean the following: Suppose the edge of R is $\{v,u\}$, with v in C_j and u in C_i.

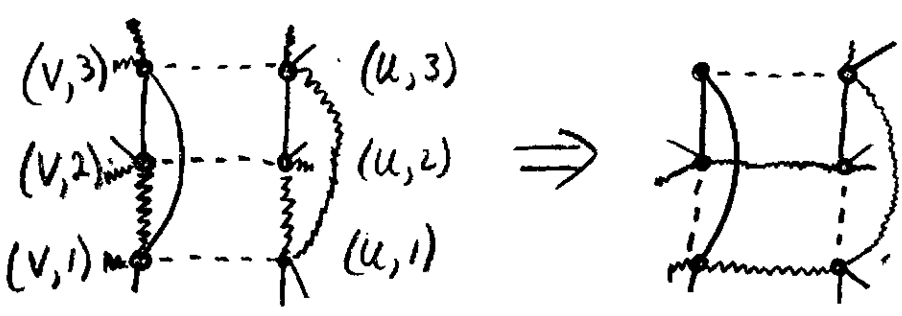

Figure 6. Facing edges $\{(v,1),(v,2)\}$, $\{(u,1),(u,2)\}$, with interchange.

The edges in the triangle (v,1), (v,2), (v,3) have already been colored red and blue, at least one being red and one being blue. Suppose {(v,a),(v,b)} is colored red. Then the "facing" edge in $C_i xC_3$ is {(u,a),(u,b)}, which must be colored red also. Similarly if C_i and C_k are joined by an edge of B, C_k preceding C_i, then $C_i xC_3$ must be colored so that a blue edge of $C_i xC_3$ faces a blue edge of $C_k xC_3$ along the edge of B. By the ordering of the circuits of F and lemma 2, this gives at most 3 constraints on the coloring of $C_i xC_3$. By lemma 1, $C_i xC_3$ can be factored into two hamilton circuits, one red, the other blue, so that the colors agree with these three (or fewer) constraints.

GxC_3 is still not colored so that the red edges form a hamilton circuit. For each edge {v,u} in R, consider a pair of red edges facing each other, say {(v,a),(v,b)} and {(u,a),(u,b)}. Exchange the colors on the 4-gon ((v,a),(u,a),(u,b),(v,b)), that is, color {(v,a),(v,b)} and {(u,a),(u,b)} white and color {(v,a),(u,a)} and {(v,b),(u,b)} red. When such exchanges are done for all the edges in R, then the red edges form a hamilton circuit of GxC_3. Similarly pairs of blue edges facing each other along edges of B can be exchanged with the white edges joining them to form a blue hamilton circuit. The white edges still form a 1-factor.

There is one situation that has not yet been handled. If a circuit of F is a 4-gon, then lemma 1 does not apply, in fact is not true. However a weakened form of lemma 1 is true. If any two vertical edges of $C_4 xC_3$ are specified, then there are two disjoint hamilton circuits in $C_4 xC_3$ such that one of the specified edges is in one of the circuits, and the other edge is in the other circuit. (The proof is left to the reader.) Any 4-gons in F, say C, is adjacent to a 2-edge cut, say

$\delta(A,B)$. One edge of $\delta(A,B)$ must be in R, the other in B. Assume the vertices of C are in A. We can color all the circuits of F whose vertices are in B first. Then CxC_3 can be colored. The remaining circuits of F, if any, can then be colored.

In summary, the blue edges form a hamilton circuit, the red edges form a hamilton circuit and the white edges form a 1-factor. qed.

Combining Theorem 2 with Kotzig's results, we get

Theorem 3. If G is a cubic graph and C is a circuit, then GxC has a 1-factorization if and only if G is bridgeless or C is not of length 3.

5. More questions

Although one of Kotzig's questions has been answered, many more questions come to mind. In the following G is a cubic graph.

Question 1. Does GxC_k, $k>3$, have a hamilton decomposition?

Since C_3 is also the complete graph on 3 vertices, K_3, we can try to generalize in a different way.

Question 2. Does GxK_k, $k>3$, have a 1-factorization?

If k is even, then the answer is yes. K_k has a 1-factorization, and the product of any regular graph G and a graph H with a 1-factorization also has a 1-factorization [Himelwright and Williamson, 1974]. If k is odd, and G has a bridge, then the answer is no. Following an argument of Kotzig for GxK_3, if G has a bridge then GxK_k has a k-edge cut $\delta(A,B)$

with A and B containing an odd number of edges. Each 1-factor of $G \times K_k$ must contain at least one edge of this cut. Hence there can be at most k pairwise disjoint 1-factors in $G \times K_k$, but a 1-factorization requires k+2 disjoint 1-factors. Thus we can restrict Question 3 to G being 2-connected, and k odd.

Question 3. Does $G \times K_k$ have a hamilton decomposition?

For cases when k is odd and G has a bridge, the answer is no. A possible lemma that might be used to answer question 3 affirmatively, using a proof similar to the one in this paper, is a positive answer to:

Question 4. Does $C_n \times K_k$ have a hamilton decomposition for k odd?

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