The Topological and Algebraic Picard Groups

by

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Abstract

The Picard group of a compact Hausdorff space is the group of isomorphism classes of line bundles over that space. The Picard group of an algebra is the group of isomorphism classes of line modules over that algebra. In this thesis, we show that in the case of $C(X)$ the algebra of continuous complex-valued functions over a compact Hausdorff space X, isomorphism classes of balanced line modules over $C(X)$ are in bijection with isomorphism classes of line bundles over X, showing the relationship between the two types of Picard groups. In the second part of this thesis, we prove that, in general, the Picard group of a finite-dimensional semisimple complex algebra is isomorphic to the symmetric group on the number of components of the algebra's Wedderburn decomposition.

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Chapter 1

Introduction

An ordinary compact Hausdorff space X is completely characterized by the commutative unital $\mathbb{C}$-algebra $C(X)$, the algebra of continuous complex-valued functions on X. The concept behind noncommutative geometry is to generalize this: by the same machinery that allows us to describe an ordinary space using only the algebra of continuous functions, we are able to take a noncommutative algebra and describe the hypothetical space that the algebra's elements "would be" functions on.

A *vector bundle* is a continuous family of vector spaces parameterized by a topological space X. In the case that the vector spaces are 1-dimensional, it is called a *line bundle*. The generalization of vector bundles to noncommutative geometry are the *finitely generated projective modules* of an algebra, which should be thought of as modules of sections of vector bundles on the topological space the algebra acts on. It is correct to call this a generalization because it is known, by Swan's theorem [Swa62], that in the classical case of $C(X)$, the algebra of continuous complex-valued functions on a compact Hausdorff space X, every balanced finitely generated projective module is isomorphic to the module of sections on a vector bundle on X.

The tensor product of two vector bundles over a base space X is obtained by taking the tensor products of the corresponding vector spaces on each point. Because the tensor product of two 1-dimensional vector spaces is a 1-dimensional vector space, the tensor product of two line bundles is itself a line bundle. The Picard group of a topological space X is the group of line bundles over that space equipped with the group operation $\otimes_{C(X)}$.

Beggs and Brzeziński [BB14] proposed a definition of *line modules* as a generalization of the line bundles to noncommutative geometry (we have restated this definition in this thesis as Definition 2.1.18), and defined the Picard group of an algebra A to be the group of line modules over that algebra equipped with the group operation $\otimes_A$. Our goal in this thesis is to examine these definitions in further detail, both to confirm that the line modules are indeed a generalization and to characterize the Picard group of certain types of algebras.

Chapter 2 is an overview of necessary definitions and pre-existing theorems needed in the latter two chapters.

In Chapter 3 we show that in the case of $C(X)$ the continuous function algebra over a topological space X, the set of balanced line modules over $C(X)$ corresponds exactly to the set of line bundles on X, by Swan's theorem. This proves that the line modules are a generalization of the line bundles, and that the topological Picard group is a normal subgroup of the algebraic Picard group. This equivalence is stated as Theorem 3.2.3.

In Chapter 4 we describe the general structure of the Picard group of a finitely generated semisimple complex algebra, in particular showing that the Picard group of any algebra that satisfies those properties is isomorphic to the symmetric group on the set of components of its Wedderburn decomposition. This result is stated in Corollary 4.4.1.

Chapter 2

Preliminaries

2.1 Preliminaries for Chapter 3

2.1.1 Modules

Definition 2.1.1. Let R be a unital ring. A *left R-module* is a set M together with

1. A binary operation $+$ on M under which M is an abelian group, and
2. An action of R on M (that is, a map $R \times M \to M$ denoted by $(r, m) \mapsto rm$.
   (a) $(r + s)m = rm + sm$, for all $r, s \in R$, $m \in M$,
   (b) $(rs)m = r(sm)$, for all $r, s \in R$, $m \in M$,
   (c) $r(m + n) = rm + rn$, for all $r \in R$, $m, n \in M$, and
   (d) $1m = m$, for all $m \in M$.

A *right R-module* has analagous properties, but for an action $M \times R \to M$ from the right, denoted as $(m, r) \mapsto mr$.

If M is both a left R-module and a right S-module for rings R and S in such a way that $r(ms) = (rm)s$ for all $r \in R$, $m \in M$, $s \in S$, we call it an *R-S-bimodule*, or, in the case that $R = S$, an R-*bimodule*. If R is commutative, then an R-bimodule is *balanced* if the left and right R-module structures coincide.

Definition 2.1.2. Let R be a ring, X a right R-module, Y a left R-module, and G an Abelian group. An R-*balanced product* is a map $\phi : X \times Y \to G$ with the following properties:

1. $\phi(x, y + y') = \phi(x, y) + \phi(x, y')$ for all $x \in X$ and $y, y' \in Y$

2. $\phi(x + x', y) = \phi(x, y) + \phi(x', y)$ for all $x, x' \in X$ and $y \in Y$

3. $\phi(xr, y) = \phi(x, ry)$ for all $x \in X$, $y \in Y$, and $r \in R$

Definition 2.1.3. For a ring R, right R-module X, and left R-module Y, the *tensor product* of X and Y over R is an Abelian group $X \otimes_R Y$ together with a balanced product $\otimes_R : X \times Y \to X \otimes_R Y$ that is universal in the following sense:
for every Abelian group G and R-balanced product $f : X \times Y \to G$, there is a unique group homomorphism $\tilde{f} : X \otimes_R Y \to G$ such that $\tilde{f} \circ \otimes_R = f$.

Remark 2.1.4. For rings R, S, and T if X is an S-R-bimodule and Y is an R-T-bimodule, the tensor product $X \otimes_R Y$ is an S-T-bimodule via the operations
$s(x \otimes_R y) = (sx) \otimes_R y$ and $(x \otimes_R y)t = x \otimes_R (yt)$ for all $x \in X$, $\in Y$, $s \in S$, and $t \in T$.

2.1.2 Vector Bundles

Definition 2.1.5. A rank n $\mathbb{C}$-*vector bundle* ξ over a topological space X consists of a topological space V, and a map $p : V \to X$, such that for all $x \in X$, the preimage $p^{-1}(x)$ (called the fiber over x) is an n-dimensional $\mathbb{C}$-vector space, and there exists a cover of X by open sets U_α equipped with homemomorphisms

$$h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n,$$

such that, for all $x \in U_\alpha$, $h_\alpha(p^{-1}(x)) = \{x\} \times \mathbb{C}^n$ and $h_\alpha|_x : p^{-1}(x) \to \{x\} \times \mathbb{C}^n$ is a vector space isomorphism. We call V the total space and X the base space of ξ.

A rank 1 vector bundle is also known as a *complex line bundle*.

Analogously we can define real vector bundles and real line bundles.

Definition 2.1.6. For a vector bundle ξ given by $p : V \to X$, a *section* of ξ is a map $s : X \to V$ continuously assigning to each $x \in X$ a vector $s(x)$ in the fiber $p^{-1}(x)$.

The set of sections $\Gamma(\xi)$ is a module over $C(X)$, the ring of complex valued continuous functions on X, with the module operations being pointwise vector addition and pointwise scalar multiplication:

$\forall s_1, s_2 \in \Gamma(\xi)$, $(s_1 + s_2) : X \to V$ given by $(s_1 + s_2)(x) = s_1(x) + s_2(x) \in p^{-1}(x)$.

$\forall a \in C(X), s_1 \in \Gamma(\xi)$, $(a \cdot s_1) : X \to V$ given by $(a \cdot s_1)(x) = a(x) \cdot s_1(x) \in p^{-1}(x)$.

Remark 2.1.7. An isomorphism between vector bundles $p_\xi : V_\xi \to X$ and $p_\zeta : V_\zeta \to X$ over the same base space X is a homeomorphism $f : V_\xi \to V_\zeta$ taking each fiber $p_\xi^{-1}(x)$ to the corresponding fiber $p_\zeta^{-1}(x)$ by a linear isomorphism.

It follows that for $s : X \to V_\xi$ a section on ξ, and $f : V_\xi \to V_\zeta$ a vector bundle isomorphism, $f \circ s : X \to V_\zeta$ is a section on ζ.

Example 2.1.8. The rank n trivial bundle over a base space X is given by $p : X \times \mathbb{C}^n \to X$ with $p(x, r) = x$.

Example 2.1.9. Let V be the quotient space $([-1, 1] \times \mathbb{R})/\sim$ under the relation $(-1, r) \sim (1, -r)$ for all $r \in \mathbb{R}$. The *Mobius bundle* is the real line bundle given by $p : V \to S^1$ with $p(\theta, r) = e^{\pi i \theta}$.

Example 2.1.10. For any surface X in $\mathbb{R}^3$, the *tangent bundle* is the real vector bundle given by a total space $\bigcup_{x \in X} \{x\} \times T_x X$, where $T_x X$ is the tangent space to X at point x, a base space X, and a map $p : \bigcup_{x \in X} \{x\} \times T_x X \to X$ with $p(x, v) = x$. Here we equip $TX \subseteq X \times \mathbb{R}^3$ with the subspace topology.

A section on the tangent bundle is a vector field on X.

Example 2.1.11. The 1-dimensional complex projective space $\mathbb{CP}^1$ is the quotient space of $\mathbb{C}^2 \setminus \{0\}$ under the relation defined by $(v_1, v_2) \sim (w_1, w_2)$ if and only if

$$(v_1, v_2) = z(w_1, w_2) = (zw_1, zw_2)$$

for some $0 \neq z \in \mathbb{C}$. We denote elements of $\mathbb{CP}^1$ by $[w_1 : w_2] = [(w_1, w_2)]$.

The *tautological line bundle* of $\mathbb{CP}^1$ is constructed of a base space $\mathbb{CP}^1$, a total space V defined to be the disjoint union of all 1-dimensional subspaces of $\mathbb{C}^2$, and a map $\pi : V \to \mathbb{CP}^1$ given by $\pi(w_1, w_2) = [w_1 : w_2]$ for all $0 \neq (w_1, w_2) \in \mathbb{C}^2$. We give $V \subseteq \mathbb{CP}^1 \times \mathbb{C}$ the subspace topology. Each distinct subspace of $\mathbb{C}^2$ has a distinct 0 point in V (due to it being a disjoint union), and to explain where π maps these 0 points, it is easiest to work out the fibers for π as we have defined it.

For any $[w_1 : w_2] \in \mathbb{CP}^1$, $\pi^{-1}([w_1 : w_2]) \cap (\mathbb{C}^2 \setminus \{0\}) = \{(v_1, v_2) \in \mathbb{C}^2 \setminus \{0\} | (v_1, v_2) \sim (w_1, w_2)\} = \{z(w_1, w_2) | 0 \neq z \in \mathbb{C}\} = \mathrm{Span}((w_1, w_2)) \setminus \{0\}$.

It is now obvious that the missing piece of our definition of π is simply that for $0_{(w_1,w_2)} \in \mathrm{Span}((w_1, w_2))$, $\pi(0_{(w_1,w_2)}) = [w_1 : w_2]$. With that the fibers are given by $\pi^{-1}([w_1 : w_2]) = \mathrm{Span}((w_1, w_2))$. These fibers are clearly 1-dimensional vector spaces.

The local trivializations are as follows.

Let $U_1 = \{[w_1 : w_2] \in \mathbb{CP}^1 | w_2 \neq 0\}$. Every $[w_1 : w_2]$ in U_1 can be expressed as $[\frac{w_1}{w_2} : 1]$, because $w_2(\frac{w_1}{w_2}, 1) = (w_1, w_2)$.

Define $h_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{C}$ by $h_1(w_1, w_2) = ([\frac{w_1}{w_2} : 1], w_2)$ for $(w_1, w_2) \neq 0$ and $h_1(0_{(w_1,w_2)}) = ([\frac{w_1}{w_2} : 1], 0)$.

For any $[w_1 : w_2] \in U_1$, $\pi^{-1}([w_1 : w_2]) = \mathrm{Span}((w_1, w_2))$. For any $z(w_1, w_2) \in$

Span$((w_1, w_2))$,

$$\begin{aligned} h_1(z(w_1, w_2)) &= h_1(zw_1, zw_2) \\ &= ([\frac{zw_1}{zw_2} : 1], zw_2) \\ &= ([\frac{w_1}{w_2} : 1], zw_2) \\ &= z([\frac{w_1}{w_2} : 1], w_2) \\ &= zh_1(w_1, w_2). \end{aligned}$$

Let $U_2 = \{[w_1 : w_2] \in \mathbb{CP}^1 | w_1 \neq 0\}$. Every $[w_1 : w_2]$ in U_2 can be expressed as $[1 : \frac{w_2}{w_1}]$, because $w_1(1, \frac{w_2}{w_1}) = (w_1, w_2)$.

Define $h_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{C}$ by $h_2(w_1, w_2) = ([1 : \frac{w_2}{w_1}], w_1)$ for $(w_1, w_2) \neq 0$ and $h_2(0_{(w_1,w_2)}) = ([1 : \frac{w_2}{w_1}], 0)$.

For any $[w_1 : w_2] \in U_2$, $\pi^{-1}([w_1 : w_2]) =$ Span$((w_1, w_2))$. For any $z(w_1, w_2) \in$ Span$((w_1, w_2))$,

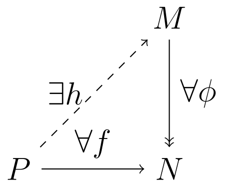
$$\begin{aligned} h_2(z(w_1, w_2)) &= h_2(zw_1, zw_2) \\ &= ([1 : \frac{zw_2}{zw_1}], zw_1) \\ &= ([1 : \frac{w_2}{w_1}], zw_1) \\ &= z([1 : \frac{w_2}{w_1}], w_1) \\ &= zh_2(w_1, w_2). \end{aligned}$$

2.1.3 Finitely Generated Projective Modules

Definition 2.1.12. An R-module P is said to be *projective* if it satisfies the following property (the lifting property):

for any R-modules M and N, surjective R-module homomorphism $\phi : M \to N$, and

R-module homomorphism $f : P \to N$, there exists an R-module homomorphism $h : P \to M$ such that $\phi \circ h = f$, so that the following diagram commutes:



Definition 2.1.13. [BB14] The *left dual* E° of a left A-module E is defined to be ${}_A\text{Hom}(E, A)$, the Abelian group of left module maps from E to A. E° has a right module structure given by $(\alpha \cdot a)(e) = \alpha(e) \cdot a$ for all $\alpha \in E^\circ$, $a \in A$, and $e \in E$.

Theorem 2.1.14. *A left A-module E is finitely generated projective if and only if there exist $e^1, ..., e^n \in E$ and $e_1, ..., e_n \in E^\circ$ such that, for all $e \in E$, $e = \sum_{i=1}^n e_i(e)e^i$. The set $\{e^i\}$ is called a* frame *and $\{e_i\}$ the corresponding* coframe.

Proof. ($\Rightarrow$): Let E be a finitely generated projective left A-module. Let X be a finite subset of E such that E is generated by X. Because X is finite it can be indexed. Let $X = \{e^1, ..., e^n\}$.

Now consider the A-module $A^n = \prod_{i=1}^n A = \{(a_1, ..., a_n) | a_i \in A\}$.

Define $g : A^n \to E$ by $g(a_1, ..., a_n) = \sum_{i=1}^n a_i e^i$. Then

$$\begin{aligned} g(A^n) &= \{g(a_1, ..., a_n) | a_i \in A\} \\ &= \{\sum_{i=1}^n a_i e^i | a_i \in A\} \\ &= \text{Span} X \\ &= E. \end{aligned}$$

Thus $g : A^n \to E$ is a surjective A-module homomorphism, and the identity map $\text{id}_E : E \to E$ is an A-module homomorphism, so by the lifting property of projective modules, there exists $h : E \to A^n$ such that $g \circ h = \text{id}_E$.

For each $1 \leq i \leq n$, define $\alpha_i : A^n \to A$ by $\alpha_i(a_1, ..., a_n) = a_i$m and let $e_i := \alpha_i \circ h : E \to A$.

As a composition of A-module homomorphisms $e_i = \alpha_i \circ h$ is an A-module homomorphism, i.e., $e_i \in {}_A\mathrm{Hom}(E, A) = E^\circ$.

Then for all $e \in E$,

$$\begin{aligned} e &= g \circ h(e) \\ &= g(\alpha_1(h(e)), ..., \alpha_n(h(e))) \\ &= g(e_1(e), ..., e_n(e)) \\ &= \sum_{i=1}^{n} e_i(e)e^i. \end{aligned}$$

This concludes the proof of ($\Rightarrow$).

($\Leftarrow$): Let E be a left A-module and suppose that there are $e^i \in E$ and $e_i \in E^\circ$ such that, for all $e \in E$, $e = \sum_{i=1}^n e_i(e)e^i$.

Let $X = \{e^1, ..., e^n\}$. This is a finite subset of E, and every element of E can be written in the form $\sum_{i=1}^n a_i e^i$ for $a_i \in A$. Therefore, E is finitely generated.

Now suppose that we have some surjective A-module homomorphism $g : P \to Q$, and some A-module homomorphism $f : E \to Q$.

For each e^i, $f(e^i) \in Q$, so because g is surjective, there exist $p^i \in P$, such that $g(p^i) = f(e^i)$.

Pick a set of $p^1, ..., p^n$ satisfying the above and use them to define $h : E \to P$ by $h(e) = \sum_{i=1}^n e_i(e)p^i$.

Now for all $d, e \in E$, and $a \in A$,

$$\begin{aligned} h(d + ae) &= \sum_{i=1}^{n} e_i(d + ae)p^i \\ &= \sum_{i=1}^{n} (e_i(d) + ae_i(e))p^i \\ &= \sum_{i=1}^{n} e_i(d)p^i + a\sum_{i=1}^{n} e_i(e)p^i \\ &= h(d) + ah(e) \end{aligned}$$

therefore $h : E \to P$ is a homomorphism.

For all $e \in E$,

$$\begin{aligned} g \circ h(e) &= g(\sum_{i=1}^{n} e_i(e)p^i) \\ &= \sum_{i=1}^{n} e_i(e)h(p^i) \\ &= \sum_{i=1}^{n} e_i(e)f(e^i) \\ &= f(\sum_{i=1}^{n} e_i(e)e^i) \\ &= f(e) \end{aligned}$$

i.e., $g \circ h = f$. This shows that E satisfies the lifting property of a projective module. Therefore E is finitely generated and projective. □

Remark 2.1.15. With $e^i \in E$ and $e_i \in E^\circ$ defined as above,

For all $\alpha \in E^\circ$, $\alpha = \sum_{i=1}^{n} e_i\alpha(e^i)$.

Proof. Let α be any element of E°.

For all $e \in E$,

$$\begin{aligned}(\sum_{i=1}^{n} e_i\alpha(e^i))(e) &= \sum_{i=1}^{n}(e_i\alpha(e^i))(e) \\ &= \sum_{i=1}^{n} e_i(e)\alpha(e^i) \\ &= \alpha(\sum_{i=1}^{n} e_i(e)e^i) \\ &= \alpha(e),\end{aligned}$$

and therefore $\alpha = \sum_{i=1}^{n} e_i\alpha(e^i)$. □

If E is a bimodule, then E° is also a bimodule, with the right-module structure as in Definition 2.1.13 and left-module structure $(a \cdot \alpha)(e) = \alpha(e \cdot a)$.

2.1.4 Line Modules

These definitions of evaluation map, coevaluation map, an line module are taken from Beggs and Brzeziński [BB14, Sections 2 and 3].

For an A-bimodule E that is finitely generated projective as a left A-module, we can define the following two bimodule maps:

Definition 2.1.16 (The Evaluation Map)**.** $\mathrm{ev} : E \otimes_A E^\circ \to A$, given by $\mathrm{ev}(e \otimes_A \alpha) = \alpha(e)$

Definition 2.1.17 (The Coevaluation Map)**.** $\mathrm{coev} : A \to E^\circ \otimes_A E$, given by $\mathrm{coev}(1_A) = \sum_{i=1}^{n} e_i \otimes_A e^i$, where $\{e^i\}$ is a frame for E and $\{e_i\}$ is its corresponding coframe.

It follows that for all $a \in A$, $\mathrm{coev}(a) = \mathrm{coev}(1_A \cdot a) = \mathrm{coev}(1_A) \cdot a = (\sum_{i=1}^{n} e_i \otimes_A e^i)a$.

Definition 2.1.18. Let E be an A-bimodule that is finitely generated as a left A-module. If $\mathrm{coev} : A \to E^\circ \otimes_A E$ is an isomorphism, then E is called a *weak left line* *module*. If in addition $\mathrm{ev} : E \otimes_A E^\circ \to A$ is an isomorphism, then E is called a *left line module*.

Definition 2.1.19. The Picard group of an algebra A, denoted $\mathrm{Pic}(A)$, is the group of isomorphism classes of line modules over A, with the operation $\otimes_A$.

2.1.5 Serre-Swan Duality

Definition 2.1.20. A topological space X is said to be *compact* if every open cover of X has a finite subcover.

Definition 2.1.21. A topological space X is a *Hausdorff space* if for any two distinct points $x, y \in X$ there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.

Theorem 2.1.22 (Swan's Theorem [Swa62])**.** *Given a compact Hausdorff space X, the category of finitely generated projective modules over the continuous function algebra $C(X)$ is equivalent to the category of finite rank vector bundles on X, where the equivalence is established by sending a vector bundle to its module of continuous sections.*

A morphism between two bundles corresponds to a homomorphism between f.g.p. modules.

Definition 2.1.23. The *Picard group* of a topological space X, denoted $\mathrm{Pic}(X)$, is the set of isomorphism classes of modules of sections of line bundles over X, with the operation $\otimes_{C(X)}$.

2.1.6 A Finite Generating Set for a Set of Sections

Remark 2.1.24. Suppose that X is a compact Hausdorff space, and ξ is a rank n vector bundle given by $\pi : V \to X$. Let $E = \Gamma(\xi)$, $A = C(X)$. A finite generating set for E as an A-module can be constructed as follows:

Let $\{U_i | i \in J\}$ be the open cover of X on which each U_i has a local trivialization, h_i. Because X is compact Hausdorff, there exists a partition of unity $\{\rho_i | i \in J\}$, $\rho_i : X \to [0,1]$ such that $\text{supp}(\rho_i) \subseteq U_i$ for each $i \in J$ and, for all $x \in X$, $\sum_{i \in J} \rho_i(x) = 1$.

On each trivial bundle $U_i \times \mathbb{C}^n$, it is easy to define a set of sections

$s_{i,1}, ..., s_{i,n} : U_i \to U_i \times \mathbb{C}^n$ where

$$s_{i,j}(x) = \left(x, \begin{bmatrix} \delta_{1j} \\ \vdots \\ \delta_{nj} \end{bmatrix} \right).$$

The set $\{s_{i,j}(x) | j \in \{1, ..., n\}\}$ generates $U_i \times \mathbb{C}^n$ as an A-module.

It follows that the set of sections of the form $h_i^{-1} \circ s_{i,j} : U_i \to \pi^{-1}(U_i)$ generates the set of sections on $\pi^{-1}(U_i)$.

Define $\sigma_{i,j} : X \to V$ by setting, for all$x \in U_i$, $\sigma_{i,j}(x) = \rho_i(x) \cdot (h_i^{-1} \circ s_{i,j})(x)$ and for all$x \notin U_i$, $\sigma_{i,j}(x) = 0 \in \pi^{-1}(x)$. This is a global section of ξ, which will be useful soon.

Let e be any element of E. For each $i \in J$ define $e_i : X \to V$ given by

$e_i(x) = \rho_i(x) \cdot e(x)$.

Each e_i is only nonzero on its support U_i, but, for all $x \in X$

$$\begin{aligned} \sum_{i \in J} e_i(x) &= \sum_{i \in J} (\rho_i(x) \cdot e(x)) \\ &= (\sum_{i \in J} \rho_i(x)) \cdot e(x) \\ &= 1 \cdot e(x) \\ &= e(x). \end{aligned}$$

The restriction $e|_{U_i} : U_i \to \pi^{-1}$ is a section on $\pi^{-1}(U_i)$, so there exist $a_{i,1}, ..., a_{i,n} \in A$

such that

$$\sum_{j=1}^{n} a_{i,j}(x) \cdot (h_i^{-1} \circ s_{i,j})(x) = e(x)$$

for all $x \in U_i$.

Now consider $\sum_{j=1}^{n} a_{i,j} \cdot \sigma_{i,j}$. For all $x \in U_i$

$$\begin{aligned}
(\sum_{j=1}^{n} a_{i,j} \cdot \sigma_{i,j})(x) &= \sum_{j=1}^{n} a_{i,j}(x) \cdot \sigma_{i,j}(x) \\
&= \sum_{j=1}^{n} a_{i,j}(x) \cdot \rho_i(x) \cdot (h_i^{-1} \circ s_{i,j})(x) \\
&= \rho_i(x) \cdot \sum_{j=1}^{n} a_{i,j}(x) \cdot (h_i^{-1} \circ s_{i,j})(x) \\
&= \rho_i(x) \cdot e(x) \\
&= e_i(x).
\end{aligned}$$

and for all$x \notin U_i$, $(\sum_{j=1}^{n} a_{i,j} \cdot \sigma_{i,j})(x) = 0$, which is equal to $e_i(x)$, so for all $x \in X$

$$\begin{aligned}
\sum_{i \in J}(\sum_{j=1}^{n} a_{i,j} \cdot \sigma_{i,j})(x) &= \sum_{i \in J} e_i(x) \\
&= e(x)
\end{aligned}$$

In other words, for every $e \in E$, there exists $\{a_{i,j} | i \in J, j \in \{1, ..., n\}\} \subseteq A$ such that $\sum_{i \in J} \sum_{j=1}^{n} a_{i,j} \cdot \sigma_{i,j} = e$. Therefore $\{\sigma_{i,j} | i \in J, j \in \{1, ..., n\}\}$ generates E as an A-module.

2.2 Preliminaries for Chapter 4

Remark 2.2.1. The standard basis of $\mathbb{C}^n$ is $\{e[n]_1, ..., e[n]_n\}$, where

$$e[n]_i = \begin{bmatrix} \delta_{1i} \\ \vdots \\ \delta_{ni} \end{bmatrix}.$$

The standard ordered basis of $M_{n_1 \times n_2}(\mathbb{C})$ is $\{e[n_1]_i e[n_2]_j^T\}$. Let $E_{i,j}[n_1 \times n_2] = e[n_1]_i e[n_2]_j^T$.

Definition 2.2.2. An $n \times n$ *permutation matrix* is a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n.

Remark 2.2.3. The group of $N \times N$ permutation matrices is isomorphic to the symmetric group S_N, with each $\sigma \in S_N$ corresponding to a matrix P_σ defined by

$$P_\sigma = \begin{bmatrix} e[N]_{\sigma(1)} & e[N]_{\sigma(2)} & \ldots & e[N]_{\sigma(N)} \end{bmatrix}.$$

For any P_σ defined this way, its inverse with respect to matrix multiplication is

$$P_\sigma^{-1} = P_\sigma^T = \begin{bmatrix} e[N]_{\sigma(1)}^T \\ e[N]_{\sigma(2)}^T \\ \vdots \\ e[N]_{\sigma(N)}^T \end{bmatrix}.$$

Proof. The fact that $P_\sigma^{-1} = P_\sigma^T$ can easily be checked explicitly:

$(P_\sigma P_\sigma^T)_{i,j} = \sum_{k=1}^N P_{\sigma,i,k} P_{\sigma,k,j}^T = \sum_{k=1}^N P_{\sigma,i,k} P_{\sigma,j,k}$.

For given i and $j = i$, $P_{\sigma,i,k} = P_{\sigma,j,k} = 1$ for $k = \sigma^{-1}(i)$ and 0 for all other $k = 1, ..., N$.

Therefore, for $i = j$, $(P_\sigma P_\sigma^T)_{i,j} = 1$.

For any i, j such that $i \neq j$, there does not exist a k such that both $P_{\sigma,i,k}$ and $P_{\sigma,j,k}$ are nonzero. Therefore, for $i \neq j$, $(P_\sigma P_\sigma^T)_{i,j} = 0$.

Having calculated every entry, we can see that $P_\sigma P_\sigma^T = I_N$. □

The following statement of Wedderburn's theorem is lifted directly from Dummit and Foote's *Abstract Algebra* [DF04], conditions 1 and 2 will not be used in this paper.

Theorem 2.2.4 (Wedderburn's Theorem [DF04, Sections 18.2, Theorem 4]). *Let R be a nonzero ring with identity. Then the following are equivalent:*

1. *Every R-module is projective*

2. *Every R-module is injective*

3. *Every R-module is completely reducible*

4. *The ring R considered as a left R-module is a direct sum:*

$$R = L_1 \oplus L_2 \oplus ... \oplus L_n,$$

where each L_i is a simple module with $L_i = Re_i$ for some $e_i \in R$ with

(i) $e_ie_j = 0$ *if* $i \neq j$,

(ii) $e_i^2 = e_i$ *for all* i,

(iii) $\sum_{i=1}^{n} e_i = 1$.

5. *As rings, R is isomorphic to a direct product of matrix rings over division rings, i.e., $R = R_1 \times R_2 \times ... \times R_r$, where R_j is a two-sided ideal of R and R_j is isomorphic to the ring of all $n_j \times n_j$ matrices with entries in a division ring Δ_j, $j = 1, 2, ..., r$. The integer r, the integers n_j, and the division rings Δ_j (up to isomorphism) are uniquely determined by R.*

Definition 2.2.5. A ring satisfying any of the equivalent properties in Wedderburn's theorem 2.2.4 is called *semisimple*.

Definition 2.2.6. The rings R_i in condition (5) of Wedderburn's Theorem are called the *Wedderburn components* and the expression of R as a direct sum of them is called the *Wedderburn decomposition*.

In the case that R is a finite dimensional semisimple $\mathbb{C}$-algebra, the division rings Δ_j must all be isomorphic to $\mathbb{C}$ [DF04, Sections 18.2, Proposition 9], so R is isomorphic to a direct product $M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times ... \times M_{n_r}(\mathbb{C})$. We will only be dealing with this case.

Chapter 3

Line Modules and Classical Line Bundles

3.1 Line Modules

Per Definition 2.1.18, an A-bimodule E is a left line module when both $\mathrm{coev} : A \to E^\circ \otimes_A E$ and $\mathrm{ev} : E \otimes_A E^\circ \to A$ are isomorphisms. In order to more easily check for surjectivity and injectivity of coev we will define two maps:

Definition 3.1.1. $\iota : E^\circ \otimes_A E \to {}_A\mathrm{Hom}(E, E)$, where $\iota(f \otimes_A \eta)(e) = f(e)\eta$

Definition 3.1.2. $\psi : {}_A\mathrm{Hom}(E, E) \to E^\circ \otimes_A E$, $\psi(T) = \sum_{i=1}^n e_i \otimes_A T(e^i)$

Remark 3.1.3. ι and ψ are inverses of each other, and therefore are both bijective.

Proof. The composition of functions $\psi \circ \iota$ maps tensors to themselves:

$$\begin{aligned}\psi \circ \iota(f \otimes_A \eta) &= \psi(\iota(f \otimes_A \eta)) \\ &= \sum_{i=1}^{n} e_i \otimes \iota(f \otimes_A \eta)(e^i) \\ &= \sum_{i=1}^{n} e_i \otimes_A f(e^i)\eta \\ &= \sum_{i=1}^{n} e_i f(e^i) \otimes_A \eta \\ &= f \otimes_A \eta,\end{aligned}$$

and the composition of functions $\iota \circ \psi$ maps homomorphisms to themselves: for all $e \in E$,

$$\begin{aligned}\iota \circ \psi(T)(e) &= \iota(\sum_{i=1}^{n} e_i \otimes_A T(e^i))(e) \\ &= \sum_{i=1}^{n} e_i(e)T(e^i) \\ &= T(\sum_{i=1}^{n} e_i(e)e^i) \\ &= T(e)\end{aligned}$$

therefore $\iota \circ \psi(T) = T$. □

Corollary 3.1.4. *The map coev is injective/surjective if and only if* $\iota \circ coev : A \to {}_A Hom(E, E)$ *is injective/surjective.*

Lemma 3.1.5. *The coevaluation map is injective if and only if for all nonzero* $a \in A$*, there exists* $e \in E$ *such that* $e \cdot a \neq 0$*.*

Proof. For $a \in A$,

$$\begin{aligned}\iota \circ \operatorname{coev}(a) &= \iota((\sum_{i=1}^{n} e_i \otimes_A e^i)a) \\ &= \iota(\sum_{i=1}^{n} e_i \otimes_A e^i a) \\ &= \sum_{i=1}^{n} \iota(e_i \otimes_A e^i a),\end{aligned}$$

so for all $e \in E$,

$$\begin{aligned}(\iota \circ \operatorname{coev}(a))(e) &= (\sum_{i=1}^{n} \iota(e_i \otimes_A e^i a))(e) \\ &= \sum_{i=1}^{n} (\iota(e_i \otimes_A e^i a))(e) \\ &= \sum_{i=1}^{n} e_i(e) e^i a \\ &= (\sum_{i=1}^{n} e_i(e) e^i) a \\ &= e \cdot a.\end{aligned}$$

This can be used to characterize $\operatorname{Ker}(\iota \circ \operatorname{coev})$:

$$\begin{aligned}\operatorname{Ker}(\iota \circ \operatorname{coev}) &= \{a \in A | \iota \circ \operatorname{coev}(a) = 0\} \\ &= \{a \in A | \forall e \in E,\ e \cdot a = 0\}\end{aligned}$$

An element of A *not* being in $\operatorname{Ker}(\iota \circ \operatorname{coev})$ is equivalent to stating that there exists $e \in E$ such that $e \cdot a \neq 0$. □

Lemma 3.1.6. *The coevaluation map is surjective if and only if every left A-module endomorphism on E is equivalent to right multiplication by some element of A.*

Proof.

$$\begin{aligned}\operatorname{Im}(\iota \circ \operatorname{coev}) &= \{T \in {}_A\operatorname{Hom}(E,E) | \exists a \in A \text{ such that } \iota \circ \operatorname{coev}(a) = T\} \\ &= \{T \in {}_A\operatorname{Hom}(E,E) | \exists a \in A \text{ such that } \forall e \in E,\, T(e) = e \cdot a\}\end{aligned}$$

□

Lemma 3.1.7. *The evaluation map is injective if and only if for all nonzero $\sum \eta_j \otimes_A f_j \in E \otimes_A E^\circ$, $\sum f_j(\eta_j) \neq 0$.*

Proof.

$$\begin{aligned}\operatorname{Ker}(\operatorname{ev}) &= \{\sum \eta_j \otimes_A f_j \in E \otimes_A E^\circ | \operatorname{ev}(\sum \eta_j \otimes_A f_j) = 0\} \\ &= \{\sum \eta_j \otimes_A f_j \in E \otimes_A E^\circ | \sum f_j(\eta_j) = 0\}\end{aligned}$$

□

Lemma 3.1.8. *The evaluation map is surjective if and only if there exists*

$$\sum \eta_j \otimes_A f_j \in E \otimes_A E^\circ$$

such that $\sum f_j(\eta_j) = 1_A$.

Proof. Suppose that there exists $\sum \eta_j \otimes_A f_j \in E \otimes_A E^\circ$ such that $\sum f_j(\eta_j) = 1_A$. Then, for all $a \in A$, $a(\sum \eta_j \otimes_A f_j) \in E \otimes_A E^\circ$ with $\operatorname{ev}(a(\sum \eta_j \otimes_A f_j)) = a \cdot \operatorname{ev}(\sum \eta_j \otimes_A f_j) = a \cdot 1_A = a$. □

Proposition 3.1.9. *If E is a weak left line A-module (recall Definition 2.1.18) then (up to natural isomorphism)*

$\operatorname{ev} \otimes_A \operatorname{id}_E = \operatorname{id}_E \otimes_A \operatorname{coev}^{-1} : E \otimes_A E^\circ \otimes_A E \to E,$ *and*

$\operatorname{id}_{E^\circ} \otimes_A \operatorname{ev} = \operatorname{coev}^{-1} \otimes_A \operatorname{id}_{E^\circ} : E^\circ \otimes_A E \otimes_A E^\circ \to E^\circ.$

Proof. For all $e \in E$, $a \in A$,

$$\begin{aligned}
(\,\mathrm{ev} \otimes_A \,\mathrm{id}_E) \circ (\,\mathrm{id}_E \otimes_A \,\mathrm{coev})(e \otimes_A a) &= (\,\mathrm{ev} \otimes_A \,\mathrm{id}_E)(\,\mathrm{id}_E(e) \otimes_A \,\mathrm{coev}(a)) \\
&= (\,\mathrm{ev} \otimes_A \,\mathrm{id}_E)(e \otimes_A (\sum_{i=1}^{n} e_i \otimes_A e^i)a) \\
&= \sum_{i=1}^{n} (\,\mathrm{ev} \otimes_A \,\mathrm{id}_E)(e \otimes_A e_i \otimes_A e^i \cdot a) \\
&= \sum_{i=1}^{n} (\,\mathrm{ev}(e \otimes_A e_i) \otimes_A \,\mathrm{id}_E(e^i \cdot a)) \\
&= \sum_{i=1}^{n} e_i(e) \otimes_A e^i \cdot a \\
&= \sum_{i=1}^{n} 1_A \otimes_A e_i(e)e^i \cdot a \\
&= 1_A \otimes_A e \cdot a,
\end{aligned}$$

while

$$\begin{aligned}
(\,\mathrm{id}_E \otimes_A \,\mathrm{coev}^{-1}) \circ (\,\mathrm{id}_E \otimes_A \,\mathrm{coev})(e \otimes_A a) &= (\,\mathrm{id}_E \otimes_A \,\mathrm{coev}^{-1})(\,\mathrm{id}_E(e) \otimes_A \,\mathrm{coev}(a)) \\
&= \,\mathrm{id}_E(e) \otimes_A \,\mathrm{coev}^{-1}(\,\mathrm{coev}(a)) \\
&= e \otimes_A a \\
&= e \cdot a \otimes_A 1_A,
\end{aligned}$$

so that

$$
\begin{aligned}
(\,\mathrm{id}_{E^\circ} \otimes_A \mathrm{ev}) \circ (\,\mathrm{coev} \otimes_A \mathrm{id}_{E^\circ})(a \otimes_A \alpha) &= (\,\mathrm{id}_{E^\circ} \otimes_A \mathrm{ev})(\,\mathrm{coev}(a) \otimes_A \mathrm{id}_{E^\circ}(\alpha)) \\
&= (\,\mathrm{id}_{E^\circ} \otimes_A \mathrm{ev})((\sum_{i=1}^{n} e_i \otimes_A e^i)a \otimes_A \alpha) \\
&= \sum_{i=1}^{n} (\,\mathrm{id}_{E^\circ} \otimes_A \mathrm{ev})(e_i \otimes_A e^i \cdot a \otimes_A \alpha) \\
&= \sum_{i=1}^{n} (\,\mathrm{id}_{E^\circ} \otimes_A \mathrm{ev})(e_i \otimes_A e^i \cdot a \otimes_A \alpha) \\
&= \sum_{i=1}^{n} (\,\mathrm{id}_{E^\circ}(e_i) \otimes_A \mathrm{ev}(e^i \cdot a \otimes_A \alpha)) \\
&= \sum_{i=1}^{n} e_i \otimes_A \alpha(e^i \cdot a) \\
&= \sum_{i=1}^{n} e_i \alpha(e^i \cdot a) \otimes_A 1_A \\
&= \sum_{i=1}^{n} e_i (a \cdot \alpha)(e^i) \otimes_A 1_A \\
&= a \cdot \alpha \otimes_A 1_A,
\end{aligned}
$$

and hence

$$
\begin{aligned}
(\,\mathrm{coev}^{-1} \otimes_A \mathrm{id}_{E^\circ}) \circ (\,\mathrm{coev} \otimes_A \mathrm{id}_{E^\circ})(a \otimes_A \alpha) &= (\,\mathrm{coev}^{-1} \otimes_A \mathrm{id}_{E^\circ})(\,\mathrm{coev}(a) \otimes_A \mathrm{id}_{E^\circ}(\alpha)) \\
&= \mathrm{coev}^{-1}(\,\mathrm{coev}(a)) \otimes_A \mathrm{id}_{E^\circ}(\alpha) \\
&= a \otimes_A \alpha \\
&= 1_A \otimes_A a \cdot \alpha.
\end{aligned}
$$

□

Proposition 3.1.10. *If E is a weak left line A-module and the evaluation map is surjective, then E is a left line A-module.*

Proof. Suppose that coev is known to be an isomorphism and ev is known to be surjective. Let $\gamma = \operatorname{Ker}(\operatorname{ev})$, let β be an element of $E \otimes_A E^\circ$ such that $\operatorname{ev}(\beta) = 1_A$. There is a natural isomorphism $\gamma \cong \gamma \otimes_A 1_A$, so $\gamma = \{0\}$ if and only if $\gamma \otimes_A 1_A = \{0\}$. With the help of Propostion 3.1.9 we have:

$$\begin{aligned}
\gamma \otimes_A 1_A &= (\operatorname{id}_E \otimes_A \operatorname{id}_{E^\circ} \otimes_A \operatorname{ev})(\gamma \otimes_A \beta) \\
&= (\operatorname{id}_E \otimes_A \operatorname{coev}^{-1} \otimes_A \operatorname{id}_{E^\circ})(\gamma \otimes_A \beta) \\
&= (\operatorname{ev} \otimes_A \operatorname{id}_E \otimes_A \operatorname{id}_{E^\circ})(\gamma \otimes_A \beta) \\
&= (\operatorname{ev})(\gamma) \otimes_A (\operatorname{id}_E \otimes_A \operatorname{id}_{E^\circ})(\beta) \\
&= \{0\} \otimes_A \beta \\
&= \{0\}.
\end{aligned}$$

Thus $\operatorname{Ker}(\operatorname{ev}) = \gamma \cong \gamma \otimes_A 1_A = \{0\}$, i.e., the evaluation map is injective. Together with our initial assumptions, this means E is a left line module. □

3.2 Serre-Swan Duality

Lemma 3.2.1. *Suppose that X is a compact Hausdorff space, and ξ is a rank n vector bundle given by $\pi : V \to X$. Let $E = \Gamma(\xi)$, $A = C(X)$. Then $\operatorname{coev} : A \to E^\circ \otimes_A E$ is injective.*

Proof. Let a be any nonzero element of A. Then there exists $x \in X$ such that $a(x) \neq 0$.

For $a(x)$ a nonzero element of $\mathbb{C}$ we can easily define an open ball $B = \mathbb{B}(a(x), \frac{|a(x)|}{2})$ and know that $0 \notin B$.

Since B is a neighborhood of $a(x)$, by continuity of a, $a^{-1}(B)$ is a neighborhood of x, and for all $y \in a^{-1}(B)$, $a(y) \neq 0$.

By definition of a vector bundle, there exists a neighborhood of x, $U_\alpha \subseteq X$, with local trivialization $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$.

Let $U = a^{-1}(B) \cap U_\alpha$, as a finite intersection of neighborhoods of x, we know that U must also be open and contain x.

For all $u \in U$, $a(u) \neq 0$

Since $U \subseteq U_\alpha$, $\pi^{-1}(U) \subseteq \pi^{-1}(U_\alpha)$, so we can define $h = h_\alpha|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U \times \mathbb{C}^n$.

Define $s : U \to U \times \mathbb{C}^n$ by

$$s(u) = \left(u, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right),$$

Then s is a section on the trivial bundle $U \times \mathbb{C}^n$.

Now h is a vector bundle isomorphism, so the composition $h^{-1} \circ s : U \to \pi^{-1}(U)$ is a local section of ξ on U.

Because X is compact Hausdorff, by Urysohn's Lemma [Rud66, Chapter 2, Lemma 2.12] there exists $\omega : X \to [0,1]$ such that $\omega(\{x\}) = \{1\}$ and $\omega(X \setminus U) = \{0\}$.

Define $\sigma : X \to V$ by, for all $y \in U$, $\sigma(y) = \omega(y) \cdot (h^{-1} \circ s)(y)$ and for all $y \notin U$, $\sigma(y) = 0 \in \pi^{-1}(y)$.

Then σ is a continuous global section of ξ, i.e., it is an element of E, and

$$\sigma(x) = \omega(x) \cdot (h^{-1} \circ s)(x) = h^{-1} \left(x, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \neq 0.$$

This implies $(\sigma \cdot a)(x) = \sigma(x) \cdot a(x) \neq 0$, and therefore $\sigma \cdot a \neq 0$.

By Lemma 3.1.5 the existence of such a $\sigma \in E$ for any nonzero $a \in A$ proves that the coevaluation map is injective. $\square$

Lemma 3.2.2. *Suppose that X is a compact Hausdorff space, and ξ is a rank n vector bundle given by $\pi : V \to X$. Let $E = \Gamma(\xi)$, $A = C(X)$. Then $\mathrm{ev} : E \otimes_A E^\circ \to A$ is surjective.*

Proof. Consider $\{U_i | i \in I\}$ the open cover of X with local trivializations that exists per the definition of a vector bundle. Because X is compact there exists a finite subset $J \subseteq I$ such that $\{U_i | i \in J\}$ is an open cover of X.

For each U_i, we can define a map $s_i : U_i \to U_i \times \mathbb{C}^n$ by

$$s_i(x) = \left(x, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right),$$

s_i is a section on the trivial bundle $U_i \times \mathbb{C}^n$.

The local trivialization $h_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^n$ is an isomorphism, so the composition $h_i^{-1} \circ s_i : U_i \to \pi^{-1}(U_i)$ is a local section of ξ on U_i.

Because X is compact Hausdorff, there exists a partition of unity $\{\tau_i | i \in J\}$, $\tau_i : X \to [0, 1]$, such that $\text{supp}(\tau_i) \subseteq U_i$ for each $i \in J$ and for all $x \in X$, $\sum_{i \in J} \tau_i(x) = 1$.

Define $\rho_i : X \to [0, 1]$ by $\rho_i(x) = \sqrt{\tau_i(x)}$.

Define $\sigma_i : X \to V$ by $\forall x \in U_i$, $\sigma_i(x) = \rho_i(x) \cdot (h_i^{-1} \circ s_i)(x)$ and $\forall x \notin U_i$, $\sigma_i(x) = 0 \in \pi^{-1}(x)$.

Each σ_i is a continuous global section on ξ, i.e., they are all elements of E.

Define $p : X \times \mathbb{C}^n \to \mathbb{C}$ by

$$p \left(x, \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = a_1$$

Define $\alpha_i : E \to A$, with $\alpha_i(e) : X \to \mathbb{C}$ defined by for all $x \in U_i$, $(\alpha_i(e))(x) = \rho_i(x) \cdot (p \circ h_i \circ e)(x)$ and for all $x \notin U_i$, $(\alpha_i(e))(x) = 0$.

Now we can work out that, for $x \in U_i$

$$
\begin{aligned}
(\operatorname{ev}(\sigma_i \otimes_A \alpha_i))(x) &= (\alpha_i(\sigma_i))(x) \\
&= \rho_i(x) \cdot (p \circ h_i \circ \sigma_i)(x) \\
&= \rho_i(x) \cdot (p \circ h_i)(\rho_i(x) \cdot (h_i^{-1} \circ s_i)(x)) \\
&= \rho_i(x) \cdot \rho_i(x) \cdot (p \circ h_i)((h_i^{-1} \circ s_i)(x)) \\
&= (\rho_i(x))^2 \cdot (p \circ h_i \circ h_i^{-1} \circ s_i)(x) \\
&= \tau_i(x) \cdot (p \circ s_i)(x) \\
&= \tau_i(x) \cdot 1 \\
&= \tau_i(x)
\end{aligned}
$$

And for $x \notin U_i$, $(\operatorname{ev}(\sigma_i \otimes_A \alpha_i))(x) = (\alpha_i(\sigma_i))(x) = 0$, which is equal to $\tau_i(x)$, so globally $(\operatorname{ev}(\sigma_i \otimes_A \alpha_i))(x) = \tau_i(x)$.

For the sum of tensors $\sum_{i \in J} \sigma_i \otimes_A \alpha_i \in E \otimes_A E^\circ$, this means that, for all $x \in X$

$$
\begin{aligned}
(\operatorname{ev}(\sum_{i \in J} \sigma_i \otimes_A \alpha_i))(x) &= \sum_{i \in J} (\operatorname{ev}(\sigma_i \otimes_A \alpha_i))(x) \\
&= \sum_{i \in J} \tau_i(x) \\
&= 1
\end{aligned}
$$

i.e., $\operatorname{ev}(\sum_{i \in J} \sigma_i \otimes_A \alpha_i) = 1_A$, which by Lemma 3.1.8 means the evaluation map is surjective. □

Theorem 3.2.3. *Given a compact Hausdorff space X, a finitely generated projective balanced $C(X)$-bimodule is a line module if and only if it is isomorphic to the module of continous sections of some line bundle over X.*

Proof. ($\Leftarrow$): Suppose that ξ is a line bundle over X given by $\pi : V \to X$. Let $A = C(X)$, and let $E = \Gamma(\xi)$. By Swan's Theorem 2.1.22 we already know that E

is a finitely generated projective module. By Lemmas 3.2.1 and 3.2.2, we already know that the coevaluation map is injective and the evaluation map is surjective, and by Proposition 3.1.10 if E is a weak left line module, it immediately follows that it is a left line module, so the one thing left to prove is that the coevaluation map is surjective.

Lemma 3.1.6 states that the coevaluation map is surjective if and only if any A-module endomorphism on E is equivalent to right multiplication by some element of A.

By definition of a line bundle, there is an open cover $\{U_i|i \in I\}$ of X with local trivializations $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$. Because X is compact there exists a finite subset $J \subseteq I$ such that $\{U_i|i \in J\}$ is still an open cover of X.

Let $T : E \to E$ be any A-module endomorphism of E.

The set of sections of $\pi^{-1}(U_i)$ is isomorphic to the set of sections of $U_i \times \mathbb{C}$ via composition with h_i, and the set of sections on $U_i \times \mathbb{C}$ is isomorphic to $C(U_i)$ via $\theta : C(U_i) \to \Gamma(U_i \times \mathbb{C})$, where $(\theta(a))(x) = (x, a(x))$.

Let $P_i : \Gamma(\pi^{-1}(U_i)) \to C(U_i)$ be given by for all $e \in \Gamma(\pi^{-1}(U_i))$, $P_i(e) = \theta^{-1}(h_i \circ e)$.

Let $Q_i = P_i \circ T \circ P_i^{-1} : C(U_i) \to C(U_i)$.

For all $a \in C(U_i)$,

$$\begin{aligned} Q_i(a) &= Q_i(a \cdot 1_A) \\ &= a \cdot Q_i(1_A) \end{aligned}$$

For $e \in \Gamma(\pi^{-1}(U_i))$,

$$\begin{aligned} T(e) &= (P_i^{-1} \circ Q_i \circ P_i)(e) \\ &= (P_i^{-1} \circ Q_i)(P_i(e)) \\ &= P_i^{-1}(Q_i(P_i(e))) \\ &= P_i^{-1}((P_i(e)) \cdot Q_i(1_A)) \\ &= P_i^{-1}(P_i(e)) \cdot Q_i(1_A) \\ &= e \cdot Q_i(1_A) \end{aligned}$$

Define $Q : X \to \mathbb{C}$ by, for each $i \in J$ and $x \in U_i$, setting $Q(x) = (Q_i(1_A))(x)$. For $x \in U_i \cap U_j$, this states both $Q(x) = (Q_i(1_A))(x)$ and $Q(x) = (Q_j(1_A))(x)$, which looks like it would be a problem, but is actually fine.

Let e be any section of ξ that does not vanish at x. Because $e(x)$ is a nonzero complex number, it has a multiplicative inverse, so we can say

$$\begin{gathered} (T(e))(x) = (T(e))(x) \\ \Rightarrow (e \cdot Q_i(1_A))(x) = (e \cdot Q_j(1_A))(x) \\ \Rightarrow e(x) \cdot Q_i(1_A)(x) = e(x) \cdot Q_j(1_A)(x) \\ \Rightarrow (e(x))^{-1} \cdot e(x) \cdot Q_i(1_A)(x) = (e(x))^{-1} \cdot e(x) \cdot Q_j(1_A)(x) \\ \Rightarrow Q_i(1_A)(x) = Q_j(1_A)(x). \end{gathered}$$

Thus, Q being well defined follows from T being well defined. And Q is continuous at every point in X because every point is within some U_i on which Q is equal to the continuous map $Q_i(1_A)$.

For arbitrary $T \in {}_A\mathrm{Hom}(E, E)$, we have found $Q \in A$ such that, for all $e \in E$, $T(e) = e \cdot Q$.

Therefore the coevaluation map is surjective, and alongside what we've already proven, this makes E a line module.

This concludes the proof of ($\Leftarrow$).

($\Rightarrow$): Suppose that B is a balanced line module over $A = C(X)$. B is a finitely generated projective module, so by Theorem 2.1.22, we already know that there exists a vector bundle ξ over X such that $B \cong \Gamma(\xi)$. Let ξ be given by $\pi : V \to X$, and let $E = \Gamma(\xi)$. By virtue of being isomorphic to B, we know that E is a line module.

Suppose by a contradiction that ξ is not a line bundle, i.e., it is rank n for some $n \neq 1$. There exists an open cover $\{U_i | i \in I\}$ of X with local trivializations $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$.

Because X is compact Hausdorff, there exists a partition of unity $\{\rho_i | i \in J\}$, $\rho_i : X \to [0, 1]$, such that $\text{supp}(\rho_i) \subseteq U_i$ for each $i \in J$ and, for all $x \in X$, $\sum_{i \in J} \rho_i(x) = 1$.

On each trivial bundle $U_\alpha \times \mathbb{C}^n$, each fiber $\{x\} \times \mathbb{C}^n$ has basis $\{(x, e_j) | j \in \{1, ..., n\}\}$ where

$$e_j = \begin{bmatrix} \delta_{1j} \\ \vdots \\ \delta_{nj} \end{bmatrix}.$$

Define a map $f : U_\alpha \times \mathbb{C}^n \to U_\alpha \times \mathbb{C}^n$ by $f(x, e_1) = (x, e_1)$ and for $j \neq 1$, $f(x, e_j) = (x, 0)$. This gives us a map $h_\alpha^{-1} \circ f \circ h_\alpha : \pi^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha)$.

Define $T : V \to V$ by, for $\pi(v) \in U_\alpha$, $T(v) = (\rho_\alpha(\pi(v))) \cdot ((h_\alpha^{-1} \circ f \circ h_\alpha)(v))$ and for $\pi(v) \notin U_\alpha$, $T(v) = 0 \in \pi^{-1}(\pi(v))$.

$\Gamma(T) : E \to E$ is given by $(\Gamma(T))(e) = T \circ e : X \to V$.

$(\Gamma(T))(e)$ is continuous because it is a composition of continuous functions.

For $x \in U_\alpha$,

$$\begin{aligned}((\Gamma(T))(e))(x) &= T(e(x)) \\ &= (\rho_\alpha(\pi(e(x)))) \cdot ((h_\alpha^{-1} \circ f \circ h_\alpha)(e(x))) \\ &= (\rho_\alpha(x)) \cdot ((h_\alpha^{-1} \circ f \circ h_\alpha \circ e)(x)) \\ &= (\rho_\alpha \cdot (h_\alpha^{-1} \circ f \circ h_\alpha \circ e))(x),\end{aligned}$$

which is an element of $\pi^{-1}(x)$ because $e(x) \in \pi^{-1}(x)$, $\rho_\alpha(x)$ is a scalar, and h_α^{-1}, f, and h_α are all fiber preserving maps.

Meanwhile for $x \notin U_\alpha$, $((\Gamma(T))(e))(x) = (T(e(x)) = 0 \in \pi^{-1}(x)$.

Thus, $(\Gamma(T))(e)$ is a continuous map from X to V such that for all $x \in X$, $((\Gamma(T))(e))(x) \in \pi^{-1}(x)$, hence it is a global section on ξ, i.e., an element of E. This proves that $\Gamma(T)$ is indeed a map from E to itself.

For all $e \in E$, $a \in A$, and $x \in X$,

$$\begin{aligned}((\Gamma(T))(a \cdot e))(x) &= T((a \cdot e)(x)) \\ &= T((a(x)) \cdot (e(x))) \\ &= (a(x)) \cdot (T(e(x))) \\ &= (a(x)) \cdot ((T \circ e)(x)) \\ &= (a(x)) \cdot (((\Gamma(T))(e))(x)) \\ &= (a \cdot ((\Gamma(T))(e)))(x)\end{aligned}$$

i.e., $(\Gamma(T))(a \cdot e) = a \cdot ((\Gamma(T))(e))$, so $\Gamma(T)$ is an A-module endomorphism on E.

Since E is a line module, this means coev is surjective, so by Lemma 3.1.6 every left A-module endomorphism on E is equivalent to right multiplication by some element of A. Let a be an element of A such that, for all $e \in E$, $e \cdot a = (\Gamma(T))(e)$.

Now consider the sections $\sigma_{\alpha,j} \in E$ as defined in Remark 2.1.24. For all $x \in X$,

$\sigma_{\alpha,j} \cdot a = (\Gamma(T))(\sigma_{\alpha,j})$.

For $x \in U_\alpha$

$$
\begin{aligned}
(\sigma_{\alpha,j} \cdot a)(x) &= ((\Gamma(T))(\sigma_{\alpha,j}))(x) \\
(\sigma_{\alpha,j}(x)) \cdot (a(x)) &= (T \circ \sigma_{\alpha,j})(x) \\
((\rho_\alpha(x)) \cdot ((h_\alpha^{-1} \circ s_{\alpha,j})(x))) \cdot (a(x)) &= (T(\sigma_{\alpha,j}(x))) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(s_{\alpha,j}(x))) \cdot (a(x)) &= (T((\rho_\alpha(x)) \cdot ((h_\alpha^{-1} \circ s_{\alpha,j})(x)))) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x)) \cdot (T((h_\alpha^{-1} \circ s_{\alpha,j})(x))) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x)) \cdot ((\rho_\alpha(x)) \cdot ((h_\alpha^{-1} \circ f \circ h_\alpha \circ h_\alpha^{-1} \circ s_{\alpha,j})(x))) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x)) \cdot (\rho_\alpha(x)) \cdot ((h_\alpha^{-1} \circ f \circ s_{\alpha,j})(x)) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x))^2 \cdot ((h_\alpha^{-1} \circ f \circ s_{\alpha,j})(x)) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(f(s_{\alpha,j}(x)))) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(f(x, e_j))).
\end{aligned}
$$

For $j \neq 1$,

$$
\begin{aligned}
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(f(x, e_j))) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(x, 0)) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (\rho_\alpha(x))^2 \cdot (x, 0) \\
(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j)) \cdot (a(x)) &= (x, 0)
\end{aligned}
$$

and for $j = 1$,

$$(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_1)) \cdot (a(x)) = (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(f(x, e_1)))$$
$$(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_1)) \cdot (a(x)) = (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(x, e_1))$$
$$(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_1)) \cdot (a(x)) = (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(s_{\alpha,1}(x)))$$
$$(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_1)) \cdot (a(x)) = (\rho_\alpha(x))^2 \cdot ((h_\alpha^{-1} \circ s_{\alpha,1})(x))$$
$$(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_1)) \cdot (a(x)) = (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(s_{\alpha,1}(x)))$$
$$(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_1)) \cdot (a(x)) = (\rho_\alpha(x))^2 \cdot (h_\alpha^{-1}(x, e_1)).$$

For x such that $\rho_\alpha(x) \neq 0$, this presents a problem, because that makes each $(\rho_\alpha(x)) \cdot (h_\alpha^{-1}(x, e_j))$ a nonzero vector, yet right multiplication by the complex number $a(x)$ gives a nonzero product for $j = 1$ and zero for $j \neq 1$. No such complex number exists.

This is a contradiction, proving that the set of sections of a vector bundle of rank *greater* than 1 *cannot* be a line module.

Therefore the vector bundle that a line module *is* equivalent to (via Swan's theorem) *must* be a line bundle. □

3.2.1 The Hopf Line Bundle

Example 3.2.4. We will now construct an example of a naturally occuring balanced line module which Serre-Swan duality applies to.

Consider the 3 sphere embedded in $\mathbb{C}^2$, $S^3 = \{v \in \mathbb{C}^2 | \|v\| = 1\}$, and the circle group $U(1) = (\{z \in \mathbb{C} | |z| = 1\}, \times)$.

Define a map $\pi : U(1) \to \mathrm{GL}(\mathbb{C}^2)$, where for all $z \in U(1)$, $\pi(z) : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by $(\pi(z))(v) = zv$ for all $v \in \mathbb{C}^2$.

For any $z \in U(1)$ and $v \in \mathbb{C}^2$, $\|(\pi(z))(v)\| = |z|\|v\| = \|v\|$.

Hence π restricts to a group action of $U(1)$ on S^3 by isometries. This action is called the *Hopf action*. The notation for this action is $z \triangleright w = zw$ or $z \triangleright (w_1, w_2) = (zw_1, zw_2)$.

There is similarly a group action of $U(1)$ on the vector space of complex valued continuous functions $C(S^3)$, given by $(z \triangleright f)(w) = f(z \triangleright w)$ for any $z \in U(1)$, $f \in C(S^3)$ and $w \in S^3$.

Define a quotient space $S^3/U(1)$ by, for $v, w \in S^3$, $v \sim w$ if and only if $v = z \triangleright w$ for some $z \in U(1)$.

Recall the 1-dimensional complex projective space $\mathbb{CP}^1$ as defined in Example 2.1.11. As in that example, we shall denote elements of $\mathbb{CP}^1$ by $[w_1 : w_2]$.

There is a homeomorphism $S^3/U(1) \simeq \mathbb{CP}^1$ given by $[(w_1, w_2)]_{U(1)} \mapsto [w_1 : w_2]$, ie. sending equivalence classes of ordered pairs of complex numbers in $S^3/U(1)$ to equivalence classes of the same ordered pairs under a different equivalence relation.

There is a homeomorphism $\mathbb{CP}^1 \simeq \mathbb{C} \sqcup \{\infty\}$ given by $[w_1 : w_2] \mapsto \frac{w_1}{w_2}$ for $w_2 \neq 0$ and $[w_1 : 0] \mapsto \infty$; here, open neighbourhoods of $\{\infty\}$ in $\mathbb{C} \sqcup \{\infty\}$ are compements of compact subsets of $\mathbb{C}$.

Note that $\mathbb{C} \sqcup \{\infty\} \simeq S^2$ by stereographic projection:

$w_R + iw_I \mapsto (\frac{2w_R}{1+w_R^2+w_I^2}, \frac{2w_I}{1+w_R^2+w_I^2}, \frac{w_R^2+w_I^2-1}{1+w_R^2+w_I^2})$ for any $w_R + iw_I \in \mathbb{C}$, and $\infty \mapsto (0,0,1)$.

Composing all of these homeomorphisms together, we get a homeomorphism

$\psi : S^3/U(1) \tilde{\to} S^2$, which gives rise to a pullback map $\psi^* : C(S^2) \tilde{\to} C(S^3/U(1))$, where $(\psi^*(f))([(w_1, w_2)]_{U(1)}) = f \circ \psi([(w_1, w_2)]_{U(1)})$, for any $f \in C(S^2)$, $[(w_1, w_2)]_{U(1)} \in S^3/U(1)$.

The n-th isotypical component of a representation π of $U(1)$ on a vector space V is $V_n = \{w \in V | \forall z \in U(1), (\pi(z))(w) = z^n w\}$. The 0-th isotypical component of π on V is thus simply $V_0 = \{w \in V | \forall z \in U(1), (\pi(z))(w) = w\} = V^{U(1)}$.

The n-th isotypical component of π on $C(S^3)$ is thus given by

$C(S^3)_n = \{f \in C(S^3) | \forall z \in U(1), \forall w \in S^3, (z \triangleright f)(w) = f(z \triangleright w) = z^n f(w)\}$.

In the case of the 0-th isotypical component, $C(S^3)_0 \simeq C(S^3/U(1))$ via the map

$(f : S^3 \to \mathbb{C}) \mapsto (g(f) : S^3/U(1) \to \mathbb{C})$, where $(g(f))([(w_1, w_2)]_{U(1)}) = f(w_1, w_1)$ for any $[(w_1, w_2)]_{U(1)} \in S^3/U(1)$.

We now have an isomorphism $g^{-1} \circ \psi^* : C(S^2) \tilde{\to} C(S^3)_0$.

$C(S^3)_1$ is a group by pointwise addition, i.e., for any $f_1, f_2 \in C(S^3)_1$,

$f_1 + f_2 : C(S^3) \to \mathbb{C}$ is defined to be $(f_1 + f_2)(w) = f_1(w) + f_2(w)$.

$f_1 + f_2$ is clearly in $C(S^3)$, and we can show that it is in $C(S^3)_1$ because, for all $z \in U(1)$ and $w \in S^3$:

$$\begin{aligned}(z \triangleright (f_1 + f_2))(w) &= (f_1 + f_2)(zw) \\ &= f_1(zw) + f_2(zw) \\ &= zf_1(w) + zf_2(w) \\ &= z(f_1(w) + f_2(w)) \\ &= z((f_1 + f_2)(w)).\end{aligned}$$

The 1-st isotypical component $C(S^3)_1$ is a $C(S^3)_0$-module by pointwise multiplication, i.e., the action of $C(S^3)_0$ on $C(S^3)_1$ is given by: for any $f \in C(S^3)_1$, $h \in C(S^3)_0$, $h \cdot f : C(S^3) \to \mathbb{C}$ is defined to be $(h \cdot f)(w) = (h(w)) \cdot (f(w))$.

It is clear that $h \cdot f$ is in $C(S^3)$, and we can show that it is in $C(S^3)_1$ because for all $z \in U(1)$ and $w \in S^3$:

$$\begin{aligned}(z \triangleright (h \cdot f))(w) &= (h \cdot f)(zw) \\ &= (h(zw)) \cdot (f(zw)) \\ &= (h(w)) \cdot (zf(w)) \\ &= z((h(w)) \cdot (f(w))) \\ &= z((h \cdot f)(w))\end{aligned}$$

Pointwise multiplication of complex valued functions is commutative, so the left and right actions are the same, ie. for any $w \in S^3$, $(h(w)) \cdot (f(w)) = (f(w)) \cdot (h(w))$. Therefore $C(S^3)_1$ is a balanced $C(S^3)_0$-bimodule.

Because $C(S^3)_0 \simeq C(S^2)$, $C(S^3)_1$ is a $C(S^2)$-bimodule by the operation defined for all $f \in C(S^3)_1$, and $h \in C(S^2)$, by $hf = (g^{-1} \circ \psi^*(h)) \cdot f$, $fh = f \cdot (g^{-1} \circ \psi^*(h))$.

Since S^2 is a compact Hausdorff space, Theorem 2.1.22 is applicable to any finitely generated projective module over $C(S^2)$. Note that $C(S^3)_1$ is finitely generated projective as a $C(S^2)$-module if and only if it is finitely generated projective as a $C(S^3)_0$-module. We will show that $C(S^3)_1$ is a finitely generated projective $C(S^3)_0$-bimodule by constructing a frame and coframe, as Theorem 2.1.14 shows that that is equivalent.

Define a map $e^1 : S^3 \to \mathbb{C}$ by $e^1(w_1, w_2) = w_1$. For all $z \in U(1)$, $e^1(zw_1, zw_2) = zw_1 = z(e^1(w_1, w_2))$, so $e^1 \in C(S^3)_1$.

Define a map $e^2 : S^3 \to \mathbb{C}$ by $e^1(w_1, w_2) = w_2$. For all $z \in U(1)$, $e^2(zw_1, zw_2) = zw_2 = z(e^2(w_1, w_2))$, so $e^2 \in C(S^3)_1$.

Define a map $e_1 : C(S^3)_1 \to C(S^3)_0$ by $e_1(f) = \bar{e^1} \cdot f$.

Then, for all $z \in U(1)$, and $w \in S^3$:

$$\begin{aligned}
(e_1(f))(zw) &= (\bar{e^1} \cdot f)(zw) \\
&= (\bar{e^1}(zw)) \cdot (f(zw)) \\
&= (\bar{z}\bar{e^1}(w)) \cdot (zf(w)) \\
&= \bar{z}z(\bar{e^1}(w)) \cdot (f(w)) \\
&= |z|(\bar{e^1} \cdot f)(w) \\
&= 1(e_1(f))(w)
\end{aligned}$$

Thus $e_1(f) \in C(S^3)_0$ for all $f \in C(S^3)_1$, therefore $e_1 \in C(S^3)_1^\circ$.

Define a map $e_2 : C(S^3)_1 \to C(S^3)_0$ by $e_2(f) = \bar{e^2} \cdot f$.

For all $z \in U(1)$ and $w \in S^3$:

$$\begin{aligned}(e_2(f))(zw) &= (\bar{e^2} \cdot f)(zw) \\ &= (\bar{e^2}(zw)) \cdot (f(zw)) \\ &= (\bar{z}\bar{e^2}(w)) \cdot (zf(w)) \\ &= \bar{z}z(\bar{e^2}(w)) \cdot (f(w)) \\ &= |z|^2(\bar{e^2} \cdot f)(w) \\ &= 1(e_2(f))(w)\end{aligned}$$

So $e_2(f) \in C(S^3)_0$ for all $f \in C(S^3)_1$, therefore $e_2 \in C(S^3)_1^\circ$.

For any $f \in C(S^3)$, $e_1(f)e^1 + e_2(f)e^2 \in C(S^3)_1$, and for all $w = (w_1, w_2) \in S^3$:

$$\begin{aligned}(e_1(f)e^1 + e_2(f)e^2)(w) &= (e_1(f)e^1)(w) + (e_2(f)e^2)(w) \\ &= ((e_1(f))(w)) \cdot (e^1(w)) + ((e_2(f))(w)) \cdot (e^2(w)) \\ &= (\bar{e^1}(w)) \cdot (f(w)) \cdot (e^1(w)) + (\bar{e^2}(w)) \cdot (f(w)) \cdot (e^2(w)) \\ &= \bar{w_1} \cdot (f(w)) \cdot w_1 + \bar{w_2} \cdot (f(w)) \cdot w_2 \\ &= (\bar{w}_1 w_1 + \bar{w}_2 w_2)(f(w)) \\ &= (|w_1|^2 + |w_2|^2)(f(w)) \\ &= \|w\|^2(f(w)) \\ &= 1(f(w))\end{aligned}$$

This shows that $\{e^1, e^2\}$ and $\{e_1, e_2\}$ are frame and coframe, therefore $C(S^3)_1$ is finitely generated projective.

The coframe of $C(S^3)_1$ as a $C(S^2)$-module can be constructed by composition of functions: $(\psi^*)^{-1} \circ g \circ e_1 : C(S^3)_1 \to C(S^2)$.

Now to prove that $C(S^3)_1$ is a line module, we will look at the evaluation and coevaluation maps. For notational convenience, let $A = C(S^3)_0$.

The evaluation map $\text{ev} : C(S^3)_1 \otimes_A C(S^3)_1^\circ \to C(S^3)_0$ is given by $\text{ev}(e \otimes_A \alpha) = \alpha(e)$

Because $\{e^1, e^2\}$ and $\{e_1, e_2\}$ generate $C(S^3)_1$ and $C(S^3)_1^\circ$ respectively,

$\{e^1 \otimes_A e_1, e^1 \otimes_A e_2, e^2 \otimes_A e_1, e^2 \otimes_A e_2\}$ generates $C(S^3)_1 \otimes_A C(S^3)_1^\circ$.

We will show that ev is surjective by proving that the identity map, 1_A is in $\text{Im}(\text{ev})$.

First we describe where it sends the tensors in the generating set.

$$\begin{aligned} \text{ev}(e^i \otimes_A e_j) &= e_j(e^i) \\ &= \bar{e^j} \cdot e^i \end{aligned}$$

$\forall w = (w_1, w_2) \in S^3$

$$\begin{aligned} (\text{ev}(e^i \otimes_A e_j))(w) &= (\bar{e^j} \cdot e^i)(w) \\ &= (\bar{e^j}(w)) \cdot (e^i(w)) \\ &= \bar{w_j} w_i \end{aligned}$$

With this we can pick out an element that maps to 1_A. Since

$$\begin{aligned} \text{ev}(e^1 \otimes_A e_1 + e^2 \otimes_A e_2) &= e_1(e^1) + e_2(e^2) \\ &= \bar{e^1} \cdot e^1 + \bar{e^2} \cdot e^2 \end{aligned}$$

for all $w = (w_1, w_2) \in S^3$,

$$\begin{aligned}
(\operatorname{ev}(e^1 \otimes_A e_1 + e^2 \otimes_A e_2))(w) &= (\bar{e^1} \cdot e^1 + \bar{e^2} \cdot e^2)(w) \\
&= \bar{w_1} w_1 + \bar{w_2} w_2 \\
&= |w_1|^2 + |w_2|^2 \\
&= \|w\|^2 \\
&= 1.
\end{aligned}$$

Therefore $\operatorname{ev}(e^1 \otimes_A e_1 + e^2 \otimes_A e_2) = 1_A$, therefore ev is surjective.

The coevaluation map $\operatorname{coev} : C(S^3)_0 \to C(S^3)_1^\circ \otimes_A C(S^3)_1$ is given by $\operatorname{coev}(1_A) = \sum_{i=1}^2 e_i \otimes_A e^i$

For $j, k \in \{1, 2\}$, $e_j(e^k) \in C(S^3)_0$ and so the coevaluation map sends it to

$$\begin{aligned}
\operatorname{coev}(e_j(e^k)) &= \operatorname{coev}(1_A \cdot (e_j(e^k))) \\
&= \operatorname{coev}(1_A) \cdot (e_j(e^k)) \\
&= (\sum_{i=1}^2 e_i \otimes_A e^i) \cdot (e_j(e^k)) \\
&= (\sum_{i=1}^2 e_i \otimes_A e^i) \cdot (\bar{e^j} \cdot e^k) \\
&= \sum_{i=1}^2 e_i \otimes_A e^i \cdot (\bar{e^j} \cdot e^k) \\
&= \sum_{i=1}^2 e_i \otimes_A (e^i \cdot \bar{e^j}) \cdot e^k \\
&= \sum_{i=1}^2 e_i \cdot (e^i \cdot \bar{e^j}) \otimes_A e^k
\end{aligned}$$

We now use the fact that pointwise multiplication of complex valued functions is commutative to simplify the expression of $e_i \cdot (e^i \cdot \bar{e^j})$.

For all $f \in C(S^3)_1$,

$$\begin{aligned}(e_i \cdot (e^i \cdot \bar{e^j}))(f) &= (e_i(f)) \cdot (e^i \cdot \bar{e^j}) \\ &= (\bar{e^i} \cdot f) \cdot (e^i \cdot \bar{e^j}) \\ &= \bar{e^j} \cdot f \cdot \bar{e^i} \cdot e^i \\ &= \bar{e^j} \cdot (f \cdot (\bar{e^i} \cdot e^i)) \\ &= e_j(f \cdot (\bar{e^i} \cdot e^i)) \\ &= ((\bar{e^i} \cdot e^i) \cdot e_j)(f)\end{aligned}$$

Therefore $e_i \cdot (e^i \cdot \bar{e^j}) = (\bar{e^i} \cdot e^i) \cdot e_j$.

$$\begin{aligned}\operatorname{coev}(e_j(e^k)) &= \sum_{i=1}^{2} e_i \cdot (e^i \cdot \bar{e^j}) \otimes_A e^k \\ &= \sum_{i=1}^{2} (\bar{e^i} \cdot e^i) \cdot e_j \otimes_A e^k \\ &= (\sum_{i=1}^{2} \bar{e^i} \cdot e^i) \cdot (e_j \otimes_A e^k) \\ &= (\bar{e^1} \cdot e^1 + \bar{e^2} \cdot e^2) \cdot (e_j \otimes_A e^k) \\ &= 1_A \cdot (e_j \otimes_A e^k) \\ &= e_j \otimes_A e^k\end{aligned}$$

$\{e^1, e^2\}$ and $\{e_1, e_2\}$ generate $C(S^3)_1$ and $C(S^3)_1^\circ$ respectively, so $C(S^3)_1^\circ \otimes_A C(S^3)_1$ is generated by $\{e_1 \otimes_A e^1, e_1 \otimes_A e^2, e_2 \otimes_A e^1, e_2 \otimes_A e^2\}$
$= \{\operatorname{coev}(e_1(e^1)), \operatorname{coev}(e_1(e^2)), \operatorname{coev}(e_2(e^1)), \operatorname{coev}(e_2(e^2))\}$.
Therefore coev is surjective.
Recall Lemma 3.1.5: the coevaluation map is injective if and only if for all nonzero $a \in C(S^3)_0$, there exists $e \in C(S^3)_1$, such that $e \cdot a \neq 0$.
Since $0 \neq a \in C(S^3)_0$, so there exists some $w = (w_1, w_2) \in S^3$ such that $a(w) \neq 0$.

On the one hand, $(e^1 \cdot a)(w) = (e^1(w)) \cdot (a(w)) = w_1 a(w)$, which only equals 0 if $w_1 = 0$.

On the other hand, $(e^2 \cdot a)(w) = (e^2(w)) \cdot (a(w)) = w_2 a(w)$, which only equals 0 if $w_2 = 0$.

Since $(0, 0) \notin S^3$, at least one of $e^1 \cdot a$ and $e^2 \cdot a$ must be a nonzero map. Therefore coev is injective.

This proves that $C(S^3)_1$ is a weak left line module, and knowing that ev is surjective, by Proposition 3.1.10 we are done.

Thus $C(S^3)_1$ is a left line $C(S^3)_0$-module, and so, because $C(S^3)_0 \simeq C(S^2)$, it is a left line $C(S^2)$-module. By Theorem 3.2.3, $C(S^3)_1$ is isomorphic to the set of sections on some line bundle over S^2.

Example 3.2.5. Another very similar example is

$$C(S^3)_{-1} = \{f \in C(S^3) | \forall z \in U(1), \forall w \in S^3, (z \triangleright f)(w) = f(z \triangleright w) = z^{-1} f(w)\}.$$

Note that $C(S^3)_{-1}$ is a group by pointwise addition, just like $(C^3)_1$.

For all $f_1, f_2 \in C(S^3)_{-1}$, $z \in U(1)$, and $w \in S^3$:

$$\begin{aligned}
(z \triangleright (f_1 + f_2))(w) &= (f_1 + f_2)(zw) \\
&= f_1(zw) + f_2(zw) \\
&= z^{-1} f_1(w) + z^{-1} f_2(w) \\
&= z^{-1}(f_1(w) + f_2(w)) \\
&= z^{-1}((f_1 + f_2)(w))
\end{aligned}$$

In fact, $C(S^3)_{-1}$ is a $C(S^3)_0$-module by pointwise multiplication, i.e., the action of $C(S^3)_0$ on $C(S^3)_{-1}$ given by: for all $f \in C(S^3)_{-1}$, $h \in C(S^3)_0$, $h \cdot f : C(S^3) \to \mathbb{C}$ is defined by $(h \cdot f)(w) = (h(w)) \cdot (f(w))$.

Note that $h \cdot f$ is clearly in $C(S^3)$, and we can show that it is in $C(S^3)_{-1}$ because, for all $z \in U(1)$ and $w \in S^3$:

$$\begin{aligned}(z \triangleright (h \cdot f))(w) &= (h \cdot f)(zw) \\ &= (h(zw)) \cdot (f(zw)) \\ &= (h(w)) \cdot (z^{-1} f(w)) \\ &= z^{-1}((h(w)) \cdot (f(w))) \\ &= z^{-1}((h \cdot f)(w))\end{aligned}$$

Pointwise multiplication of complex valued functions is commutative, so the left and right actions are the same, i.e., for all $w \in S^3$, $(h(w)) \cdot (f(w)) = (f(w)) \cdot (h(w))$. Therefore $C(S^3)_{-1}$ is a balanced $C(S^3)_0$-bimodule.

The proof that $C(S^3)_{-1}$ is a line module follows nearly identical logic to the proof for $C(S^3)_1$, but the choice of frame and coframe needed for this proof is as follows:

Define a map $e^1 : S^3 \to \mathbb{C}$ by $e^1(w_1, w_2) = \bar{w}_1$. For all $z \in U(1)$, $e^1(zw_1, zw_2) = z\bar{w}_1 = \bar{z}\bar{w}_1 = z^{-1}\bar{w}_1 = z^{-1}(e^1(w_1, w_2))$, so $e^1 \in C(S^3)_{-1}$.

Define a map $e^2 : S^3 \to \mathbb{C}$ by $e^1(w_1, w_2) = \bar{w}_2$. For all $z \in U(1)$, $e^2(zw_1, zw_2) = z\bar{w}_2 = \bar{z}\bar{w}_2 = z^{-1}\bar{w}_2 = z^{-1}(e^2(w_1, w_2))$, so $e^2 \in C(S^3)_{-1}$.

Define a map $e_1 : C(S^3)_{-1} \to C(S^3)_0$ by $e_1(f) = \bar{e^1} \cdot f$.

For all $z \in U(1)$ and $\forall w \in S^3$:

$$
\begin{aligned}
(e_1(f))(zw) &= (\bar{e}^1 \cdot f)(zw) \\
&= (\bar{e}^1(zw)) \cdot (f(zw)) \\
&= (\bar{z}^{-1}\bar{e}^1(w)) \cdot (z^{-1}f(w)) \\
&= \bar{z}^{-1}z^{-1}(\bar{e}^1(w)) \cdot (f(w)) \\
&= |z^{-1}|(\bar{e}^1 \cdot f)(w) \\
&= 1(e_1(f))(w)
\end{aligned}
$$

Thus, $e_1(f) \in C(S^3)_0$ for all $f \in C(S^3)_{-1}$, therefore $e_1 \in C(S^3)^{\circ}_{-1}$.

Define a map $e_2 : C(S^3)_{-1} \to C(S^3)_0$ by $e_2(f) = \bar{e}^2 \cdot f$.

For all $z \in U(1)$ and $\forall w \in S^3$:

$$
\begin{aligned}
(e_2(f))(zw) &= (\bar{e}^2 \cdot f)(zw) \\
&= (\bar{e}^2(zw)) \cdot (f(zw)) \\
&= (\bar{z}^{-1}\bar{e}^2(w)) \cdot (z^{-1}f(w)) \\
&= \bar{z}^{-1}z^{-1}(\bar{e}^2(w)) \cdot (f(w)) \\
&= |z^{-1}|^2(\bar{e}^2 \cdot f)(w) \\
&= 1(e_2(f))(w)
\end{aligned}
$$

Thus, $e_2(f) \in C(S^3)_0$ for all $f \in C(S^3)_{-1}$, therefore $e_2 \in C(S^3)^{\circ}_{-1}$.

For all $f \in C(S^3)$, $e_1(f)e^1 + e_2(f)e^2 \in C(S^3)_{-1}$, and for all $w = (w_1, w_2) \in S^3$:

$$\begin{aligned}
(e_1(f)e^1 + e_2(f)e^2)(w) &= (e_1(f)e^1)(w) + (e_2(f)e^2)(w) \\
&= ((e_1(f))(w)) \cdot (e^1(w)) + ((e_2(f))(w)) \cdot (e^2(w)) \\
&= (\bar{e^1}(w)) \cdot (f(w)) \cdot (e^1(w)) + (\bar{e^2}(w)) \cdot (f(w)) \cdot (e^2(w)) \\
&= w_1 \cdot (f(w)) \cdot \bar{w_1} + w_2 \cdot (f(w)) \cdot \bar{w_2} \\
&= (w_1\bar{w_1} + w_2\bar{w_2})(f(w)) \\
&= (|w_1|^2 + |w_2|^2)(f(w)) \\
&= \|w\|^2(f(w)) \\
&= 1(f(w))
\end{aligned}$$

This shows that $\{e^1, e^2\}$ and $\{e_1, e_2\}$ are frame and coframe for $C(S^3)_{-1}$.

With this frame and coframe, all of the logic from Example 3.2.4 can be followed without any alterations to prove that $C(S^3)_{-1}$ is a left line $C(S^3)_0$-module, and by extension a left line $C(S^2)$-module. Therefore by Theorem 3.2.3, $C(S^3)_{-1}$ is isomorphic to the set of sections on some line bundle over S^2.

Chapter 4

The Picard Group of a Semisimple Complex Algebra

4.1 Semisimplicity

Definition 4.1.1. Let A, B be unital algebras over $\mathbb{C}$. An A-B-bimodule E is irreducible (or simple) if and only if there exists no proper A-B-subbimodule.

Proposition 4.1.2. *If A and B are finite dimensional semisimple as $\mathbb{C}$-algebras then every finite dimensional A-B-bimodule is semisimple (a direct sum of irreducible subbimodules).*

Proof. Let A and B be finite dimensional semisimple $\mathbb{C}$-algebras.

Any finite dimensional A-B-bimodule E is also a left $A \otimes B^o$-module under the operation $(a \otimes b^o) \cdot e = a \cdot e \cdot b$. Clearly any subbimodule is a left submodule and vce versa, and it follows from that that simple A-B-bimodules are simple left $A \otimes B^o$-modules.

Given Wedderburn decompositions $A = \oplus_{i=1}^{N_A} M_{k_{A,i}}(\mathbb{C})$ and $B = \oplus_{j=1}^{N_B} M_{k_{B,j}}(\mathbb{C})$, we find that

$$\begin{aligned}
A \otimes B^o &= (\oplus_{i=1}^{N_A} M_{k_{A,i}}(\mathbb{C})) \otimes (\oplus_{j=1}^{N_B} M_{k_{B,j}}(\mathbb{C}))^o \\
&\simeq (\oplus_{i=1}^{N_A} M_{k_{A,i}}(\mathbb{C})) \otimes (\oplus_{j=1}^{N_B} M_{k_{B,j}}(\mathbb{C})^o) \\
&\simeq \oplus_{i=1}^{N_A} \oplus_{j=1}^{N_B} (M_{k_{A,i}}(\mathbb{C}) \otimes M_{k_{B,j}}(\mathbb{C})^o) \\
&\simeq \oplus_{i=1}^{N_A} \oplus_{j=1}^{N_B} (M_{k_{A,i}}(\mathbb{C}) \otimes M_{k_{B,j}}(\mathbb{C})) \\
&\simeq \oplus_{i=1}^{N_A} \oplus_{j=1}^{N_B} M_{k_{A,i}k_{B,j}}(\mathbb{C}).
\end{aligned}$$

This is the Wedderburn decomposition of $A \otimes B^o$. Its existence proves that $A \otimes B^o$ is semisimple.

Any finite dimensional A-B-bimodule E, by virtue of being a left $A \otimes B^o$-module, must be a direct sum of simple left $A \otimes B^o$-modules, which will of course also be simple A-B-bimodules. Therefore, E is a direct sum of irreducible subbimodules. □

Proposition 4.1.3. *If $A = \oplus_{i=1}^{N_A} M_{k_{A,i}}(\mathbb{C})$ and $B = \oplus_{j=1}^{N_B} M_{k_{B,j}}(\mathbb{C})$, then any simple A-B-bimodule is isomophic to a module of the form $M_{k_{A,i} \times k_{B,j}}(\mathbb{C})$ for some $i = 1, 2, ..., N_A$ and $j = 1, 2, ..., N_B$.*

Proof. Let E be any simple A-B-bimodule. Then E must also be simple as a left $A \otimes B^o$-module.

Continuing from the proof of 4.1.2, the components of the Wedderburn decomposition of $A \otimes B^o$ are $M_{k_{A,i}}(\mathbb{C}) \otimes M_{k_{B,j}}(\mathbb{C}) \simeq M_{k_{A,i}k_{B,j}}(\mathbb{C})$, so any simple left $A \otimes B^o$ module E is isomorphic to $\mathbb{C}^{k_{A,i}k_{B,j}}$ for some $i = 1, 2, ..., N_A$ and $j = 1, 2, ..., N_B$.

It follows that E is a $k_{A,i}k_{B,j}$-dimensional $\mathbb{C}$-vector space that $A \otimes B^o$ acts on by elements in $M_{k_{A,i}}(\mathbb{C})$ multiplying from the left and elements of $M_{k_{B,j}}(\mathbb{C})$ multiplying from the right. Therefore $E \simeq M_{k_{A,i} \times k_{B,j}}(\mathbb{C})$. □

Corollary 4.1.4. *For $A = \oplus_{i=1}^{N_A} M_{k_{A,i}}(\mathbb{C})$ and $B = \oplus_{j=1}^{N_B} M_{k_{B,j}}(\mathbb{C})$, any finite-dimensional A-B-bimodule E must be isomorphic to a module of the form $\oplus_{i=1}^{N_A} \oplus_{j=1}^{N_B} M_{k_{A,i} \times k_{B,j}}(\mathbb{C})^{m_{i,j}}$.*

In particular if $B = A$, ie. E is just an A-bimodule, then $E \simeq \oplus_{i,j=1}^{N_A} M_{k_{A,i} \times k_{A,j}}(\mathbb{C})^{m_{i,j}}$. Up to isomorphism, E is defined uniquely by the exponents $m_{i,j}$.

Theorem 4.1.5. *Given an algebra $A = \oplus_{i=1}^{N} M_{k_i}(\mathbb{C})$, there is a monoid isomorphism between $N \times N$ matrices of nonnegative integers and isomorphism classes of finite-dimensional A-bimodules, via the map $M : Bimod(A) \to M_N(\mathbb{Z}_{\geq 0})$ given by*

$$M(\oplus_{i,j=1}^{N} M_{k_i \times k_j}(\mathbb{C})^{m_{i,j}}) = \begin{bmatrix} m_{1,1} & m_{1,2} & ... & m_{1,N} \\ m_{2,1} & m_{2,2} & ... & m_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ m_{N,1} & m_{N,2} & ... & m_{N,N} \end{bmatrix}$$

*For an A-bimodule E, $M(E)$ is called the **multiplicity matrix** of E.*

Proof. By Corollary 4.1.4 we know that each isomorphism class of A-bimodules is described uniquely by the exponents that make up the entries in the multiplicity matrix. It remains to be shown that M preserves the monoid operations of Bimod(A) and $M_N(\mathbb{Z}_{\geq 0})$.

Consider A-bimodules $E_1 = \oplus_{i,j=1}^{N}(M_{k_i \times k_j}(\mathbb{C}))^{m_{1,i,j}}$ and $E_2 = \oplus_{p,q=1}^{N}(M_{k_p \times k_q}(\mathbb{C}))^{m_{2,p,q}}$. First,

$$\begin{aligned} E_1 \otimes_A E_2 &= (\oplus_{i,j=1}^{N}(M_{k_i \times k_j}(\mathbb{C}))^{m_{1,i,j}}) \otimes_A (\oplus_{p,q=1}^{N}(M_{k_p \times k_q}(\mathbb{C}))^{m_{2,p,q}}) \\ &\simeq \oplus_{i,j,p,q=1}^{N}((M_{k_i \times k_j}(\mathbb{C}))^{m_{1,i,j}} \otimes_A (M_{k_p \times k_q}(\mathbb{C}))^{m_{2,p,q}}) \\ &\simeq \oplus_{i,j,p,q=1}^{N}((M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_p \times k_q}(\mathbb{C})))^{m_{1,i,j} m_{2,p,q}}. \end{aligned}$$

For any tensor product of matrices $X \otimes_A Y \in (M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_p \times k_q}(\mathbb{C}))$, since $Y \in M_{k_p \times k_q}(\mathbb{C})$, $Y = Z_p Y$ where $Z_p = 0 \oplus ... \oplus 0 \oplus I_{k_p} \oplus 0 \oplus ... \oplus 0 \in A$.

If $j \neq p$, then $XZ_p = 0$ so

$$X \otimes_A Y = X \otimes_A Z_p Y$$
$$= XZ_p \otimes_A Y$$
$$= 0 \otimes_A Y$$
$$= 0 \otimes_A 0.$$

This means we can drop any components where $j \neq p$. So $E_1 \otimes_A E_2$ simplifies to $\oplus_{i,j,q=1}^{N}((M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C})))^{m_{1,i,j}m_{2,j,q}}$.

Given $i, j, q \in \{1, 2, ..., N\}$, consider the map $f : (M_{k_i \times k_j}(\mathbb{C})) \times (M_{k_j \times k_q}(\mathbb{C})) \to M_{k_i \times k_q}(\mathbb{C})$ given by $f(X, Y) = XY$ for $X \in M_{k_i \times k_j}(\mathbb{C})$ and $Y \in M_{k_j \times k_q}(\mathbb{C})$.

For any $X, X' \in M_{k_i \times k_j}(\mathbb{C})$, $Y, Y' \in M_{k_j \times k_q}(\mathbb{C})$, and $Z \in A$:

$f(X, Y + Y') = X \cdot (Y + Y') = XY + XY' = f(X, Y) + f(X, Y')$

$f(X + X', Y) = (X + X') \cdot Y = XY + X'Y = f(X, Y) + f(X', Y)$

$f(XZ, Y) = (XZ) \cdot Y = X \cdot (ZY) = f(X, ZY)$

This means that f is an A-balanced product. Associativity holds here because in both products XZ and ZY the only component of Z that has an effect is the jth component, so Z can be treated as just a $k_j \times k_j$ matrix.

By the universal property of the tensor product there exists a unique

$$\tilde{f} : (M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C})) \to M_{k_i \times k_q}(\mathbb{C})$$

such that $\tilde{f} \circ \otimes_A = f$. In fact this map is simply $\tilde{f}(X \otimes_A Y) = XY$.

Recall $\{E_{i,j}[n_1 \times n_2]\}$, the standard ordered basis for $M_{n_1 \times n_2}(\mathbb{C})$ from Remark 2.2.1.

Define $h : M_{k_i \times k_q}(\mathbb{C}) \to (M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C}))$ by

$$h(E_{a,b}[k_i \times k_q]) = E_{a,1}[k_i \times k_j] \otimes_A E_{1,b}[k_j \times k_q]$$

As we have defined h, it is clear that h is a $\mathbb{C}$-linear map, but it is not clear that it is an A-bimodule map. Fortunately, we do not need to know in advance that h is an A-bimodule map to show that it is the inverse of $\tilde{f}$.

Clearly, $\tilde{f} \circ h(E_{a,b}[k_i \times k_q]) = \tilde{f}(E_{a,1}[k_i \times k_j] \otimes_A E_{1,b}[k_j \times k_q]) = E_{a,1}[k_i \times k_j] E_{1,b}[k_j \times k_q] = E_{a,b}[k_i \times k_q]$. Since every element of $M_{k_i \times k_q}(\mathbb{C})$ is a linear combination of matrices of the form $E_{a,b}[k_i \times k_q]$, this means that $\tilde{f} \circ h(X) = X$ in general.

Every element of $(M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C}))$ is a linear combination of tensors of the form $E_{a,b}[k_i \times k_j] \otimes_A E_{c,d}[k_j \times k_q]$. Note that

$$\begin{aligned} E_{a,b}[k_i \times k_j] \otimes_A E_{c,d}[k_j \times k_q] &= (E_{a,1}[k_i \times k_j] E_{1,b}[k_j \times k_j]) \otimes_A (E_{c,1}[k_j \times k_j] E_{1,d}[k_j \times k_q]) \\ &= (E_{a,1}[k_i \times k_j]) \otimes_A (E_{1,b}[k_j \times k_j] E_{c,1}[k_j \times k_j] E_{1,d}[k_j \times k_q]). \end{aligned}$$

If $b \neq c$ then $E_{1,b}[k_j \times k_j] E_{c,1}[k_j \times k_j] = 0$, and if $b = c$ then $E_{1,b}[k_j \times k_j] E_{c,1}[k_j \times k_j] E_{1,d}[k_j \times k_q] = E_{1,d}[k_j \times k_q]$. That means that every element of $(M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C}))$ is a linear combination of tensors of the form $E_{a,1}[k_i \times k_j] \otimes_A E_{1,d}[k_j \times k_q]$. Finally,

$$\begin{aligned} h \circ \tilde{f}(E_{a,1}[k_i \times k_j] \otimes_A E_{1,d}[k_j \times k_q]) &= h(E_{a,1}[k_i \times k_j] E_{1,d}[k_j \times k_q]) \\ &= E_{a,1}[k_i \times k_j] \otimes_A E_{1,d}[k_j \times k_q]. \end{aligned}$$

This means that $h \circ \tilde{f}(X \otimes_A Y) = X \otimes_A Y$ in general.

Thus h is the inverse of $\tilde{f}$.

$\tilde{f}$ is a bijective A-bimodule map, so $(M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C})) \simeq M_{k_i \times k_q}(\mathbb{C})$ as A-bimodules.

At last,

$$E_1 \otimes_A E_2 \simeq \oplus_{i,j,q=1}^{N} ((M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C})))^{m_{1,i,j} m_{2,j,q}}$$
$$\simeq \oplus_{i,j,q=1}^{N} (M_{k_i \times k_q}(\mathbb{C}))^{m_{1,i,j} m_{2,j,q}}.$$

From this, we can tell that for any pair of A-bimodules $E_1 = \oplus_{i,j=1}^{N} (M_{k_i \times k_j}(\mathbb{C}))^{m_{1,i,j}}$ and $E_2 = \oplus_{p,q=1}^{N} (M_{k_p \times k_q}(\mathbb{C}))^{m_{2,p,q}}$ the entries of the multiplicity matrix of $E_3 = E_1 \otimes_A E_2$ will be $m_{3,i,q} = \sum_{j=1}^{N} m_{1,i,j} m_{2,j,q}$. In other words, the (i,q)-entry of $M(E_1 \otimes_A E_2)$ is the ith row of $M(E_1)$ multiplied with the qth column of $M(E_2)$. This is exactly the same as the matrix product $M(E_1) \cdot M(E_2)$.

Thus $M(E_1 \otimes_A E_2) = M(E_1) \cdot M(E_2)$, so $(\mathrm{Bimod}(A), \otimes_A)$ and $(M_N(\mathbb{Z}_{\geq 0}), \cdot)$ are isomorphic as monoids. □

For a monoid $(X, *)$, let $\mathrm{Inv}(X)$ be the set of invertible elements of X, equipped with the same binary operation $*$. $\mathrm{Inv}(X)$ is both a submonoid of X and a group.

If $E \in \mathrm{Bimod}(A)$ is invertible with respect to the balanced tensor product then $M(E)$ has inverse matrix $(M(E))^{-1} = M(E^{-1}) \in M(\mathrm{Bimod}(A)) = M_N(\mathbb{Z}_{\geq 0})$. In other words if $E \in \mathrm{Inv}(\mathrm{Bimod}(A))$ then $M(E) \in \mathrm{Inv}(M_N(\mathbb{Z}_{\geq 0}))$. Identical logic applies in the opposite direction to say that if $X \in \mathrm{Inv}(M_N(\mathbb{Z}_{\geq 0}))$ then $M^{-1}(X) \in \mathrm{Inv}(\mathrm{Bimod}(A))$.

Therefore M restricts to a group isomorphism $M : \mathrm{Inv}(\mathrm{Bimod}(A)) \to \mathrm{Inv}(M_N(\mathbb{Z}_{\geq 0}))$.

4.2 Identifying Inv$(M_N(\mathbb{Z}_{\geq 0}))$

Theorem 4.2.1. *The elements of Inv$(M_N(\mathbb{Z}_{\geq 0}))$ are precisely the $N \times N$ permutation matrices, i.e., Inv$(M_N(\mathbb{Z}_{\geq 0})) \simeq S_N$ per Remark 2.2.3.*

Proof. ($\Leftarrow$)The fact that the permutation matrices are non-negative integer valued is obvious from the fact that all entries are either 0 or 1, and the inverse of a permutation matrix P_σ is known to be

$$P_\sigma^{-1} = P_\sigma^T = \begin{bmatrix} e[N]_{\sigma(1)}^T \\ e[N]_{\sigma(2)}^T \\ \vdots \\ e[N]_{\sigma(N)}^T \end{bmatrix}$$

so clearly $P_\sigma \in \mathrm{Inv}(M_N(\mathbb{Z}_{\geq 0}))$.

This concludes that proof of ($\Leftarrow$).

($\Rightarrow$) By [DR14, Theorem 5.1], a matrix and its inverse are nonnegative matrices if and only if it is the product of a diagonal matrix with all positive diagonal entries and a permutation matrix. In this case, $M(E) = DP_\sigma$ and $M(E)^{-1} = P_\sigma^{-1}D^{-1}$, for some diagonal D with positive diagonal entires and P_σ a permutation matrix.

$$D = \begin{bmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & d_N \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & ... & 0 \\ 0 & d_2^{-1} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & d_N^{-1} \end{bmatrix}$$

$De[N]_i = d_i e[N]_i$,

$$M(E) = DP_\sigma = \begin{bmatrix} d_{\sigma(1)}e[N]_{\sigma(1)} & d_{\sigma(2)}e[N]_{\sigma(2)} & ... & d_{\sigma(N)}e[N]_{\sigma(N)} \end{bmatrix}$$

$e_[N]_i^T D^{-1} = e_[N]_i^T d_i^{-1}$

$$M(E)^{-1} = P_\sigma^{-1}D^{-1} = \begin{bmatrix} e[N]_{\sigma(1)}^T d_{\sigma(1)}^{-1} \\ e[N]_{\sigma(2)}^T d_{\sigma(2)}^{-1} \\ \vdots \\ e[N]_{\sigma(N)}^T d_{\sigma(N)}^{-1} \end{bmatrix}$$

If $M(E)$ and $M(E)^{-1}$ are both integer valued matrixes, that implies d_i and d_i^{-1} are both integers for all $i = 1, 2, ..., N$. This means that each $d_i = d_i^{-1} = 1$, i.e., $D = I_N$.

Therefore $M(E) = P_\sigma$. □

4.3 Identifying Inv(Bimod(A))

Theorem 4.3.1. *The elements of Inv(Bimod(A)) are precisely the isomorphism classes of left line modules of A, i.e., Inv(Bimod(A)) = Pic(A) per Definition 2.1.19.*

Proof. ($\Leftarrow$) Let E be a left line module of A, then $\mathrm{ev} : E \otimes_A E^\circ \to A$ and $\mathrm{coev} : A \to E^\circ \otimes_A E$ are bimodule isomorphisms. This means $E \otimes_A E^\circ \simeq A \simeq E^\circ \otimes_A E$. Since A is the identity element with respect to the A-balanced tensor product, E° is the inverse of E. Therefore all left line modules of A are invertible.

($\Rightarrow$) Let $[E]$ be any element of Inv(Bimod(A)). There exists an F such that $E \otimes_A F \simeq A \simeq F \otimes_A E$. The first step to proving that E is a line module will be proving that the inverse we know it has must also be isomorphic to E°.

If E is invertible, then its multiplicity matrix is a permutation matrix, and the inverse of a permutation matrix is its transpose.

For invertible $E = \oplus_{i,j=1}^N M_{k_i \times k_j}(\mathbb{C})^{m_{i,j}}$, the inverse F is the bimodule whose multiplicity matrix is the transpose of $M(E)$, ie. $F = \oplus_{i,j=1}^N M_{k_i \times k_j}(\mathbb{C})^{m_{j,i}}$, which can equivalently be written as $F = \oplus_{i,j=1}^N M_{k_j \times k_i}(\mathbb{C})^{m_{i,j}}$.

For $M(E)$ a permutation matrix, each $m_{i,j}$ is either 0 or 1, so there is no i and j such that E contains multiple copies of $M_{k_i \times k_j}(\mathbb{C})$. And there are exactly N nonzero entries of an $N \times N$ permutation matrix, so E is a direct sum of N distinct matrix modules.

Also true of permutation matrices is the fact that for each $i \in \{1, ..., N\}$ there is exactly one j such that $m_{i,j} = 1$. For a given permutation matrix, we can then re-index in terms of i. For each i let $l_i = k_j$ where $m_{i,j} = 1$. We can rewrite the direct sums as $E = \oplus_{i=1}^N M_{k_i \times l_i}(\mathbb{C})$ and $F = \oplus_{i=1}^N M_{l_i \times k_i}(\mathbb{C})$, to make it clear that they are direct sums of only N modules.

For each $i \in \{1, ..., N\}$, and any $p \in \{1, ..., l_i\}$ let $e^{i,p}$ be the element of E with $E_{1,p}[k_i \times l_i]$ in the $M_{k_i \times l_i}(\mathbb{C})$ component of E and 0 in all other components. When $e^{i,p}$ is multiplied by any $a \in A$ from the left, the result isolates the first column of the $M_{k_i}(\mathbb{C})$ component of a and puts it in the pth column of the $M_{k_i \times l_i}(\mathbb{C})$ component of E.

Define a map $\phi : F \to E^\circ$ as follows, for $c_i \in M_{l_i \times k_i}(\mathbb{C})$, $\phi(c_i) : M_{k_i \times l_i}(\mathbb{C}) \to M_{k_i}(\mathbb{C})$ is given by $(\phi(c_i))(b_i) = b_i c_i$. Each $\phi(c_i)$ is left A-linear since $(\phi(c_i))(a_i b_i) = (a_i b_i)c_i = a_i(b_i c_i) = a_i(\phi(c_i))(b_i)$. For $c = (c_1, c_2, ..., c_N)$ with each $c_i \in M_{l_i \times k_i}(\mathbb{C})$, $\phi(c)$ will be a direct sum of N distinct A-linear maps, which means it will be an A-linear map between direct sums of modules $\phi(c) : \oplus_{i=1}^N M_{k_i \times l_i}(\mathbb{C}) \to \oplus_{i=1}^N M_{k_i}(\mathbb{C})$. This shows that $\phi(c) \in E^\circ$ for any $c \in F$. Thus ϕ is well defined.

For any $a = (a_1, ..., a_N) \in A$ and $c = (c_1, ..., c_N) \in F$ the right multiplication on F is preserved by ϕ, as shown by the fact that for any $b = (b_1, ..., b_N) \in E$:

$$
\begin{aligned}
(\phi(ca))(b) &= (\phi(c_1 a_1, ..., c_N a_N))(b_1, ..., b_N) \\
&= (b_1, ..., b_N)(c_1 a_1, ..., c_N a_N) \\
&= (b_1 c_1 a_1, ..., b_N c_N a_N) \\
&= (b_1 c_1, ..., b_N c_N)(a_1, ..., a_N) \\
&= (bc)(a) \\
&= ((\phi(c))(b))(a) \\
&= ((\phi(c))a)(b)
\end{aligned}
$$

For any $a = (a_1, ..., a_N) \in A$ and $c = (c_1, ..., c_N) \in F$ the left multiplication on F is preserved by ϕ, which is again shown by how $\phi(ac)$ operates on any $b = (b_1, ..., b_N) \in E$, but it's a little bit more complicated to write.

Let $d_i = a_j \in M_{k_j}(\mathbb{C}) = M_{l_i}(\mathbb{C})$, then:

$$\begin{aligned}(\phi(ac))(b) &= (\phi(d_1c_1, ..., d_Nc_N))(b_1, ..., b_N)\\ &= (b_1, ..., b_N)(d_1c_1, ..., d_Nc_N)\\ &= (b_1d_1c_1, ..., b_Nd_Nc_N)\\ &= (b_1d_1, ..., b_Nd_N)(c_1, ..., c_N)\\ &= (ba)(c)\\ &= (\phi(c))(ba)\\ &= (a(\phi(c)))(b)\end{aligned}$$

This shows that ϕ is an A-bimodule map from F to E°.

Let c be any nonzero element of F, and let $c_i \in M_{l_i \times k_i}(\mathbb{C})$ be some nonzero component of c. For each $p \in \{1, ..., l_i\}$, $e^{i,p}c = e^{i,p}c_i$ which resides in the $M_{k_i}(\mathbb{C})$ component of A and has a first row equal to the pth row of c_i and zero in all other rows. Because c_i is a nonzero matrix, there must be some row p that has at least one nonzero entry and therefore an $e^{i,p}$ such that $e^{i,p}c \neq 0$. This means that for any $0 \neq c \in F$, $\phi(c)$ is not the zero map. Therefore ϕ is injective.

The only step left to prove that $\phi : F \to E^\circ$ is an A-bimodule isomorphism (and therefore that E° is the inverse of E) is surjectivity. This will come from showing that E° is spanned by a set of elements in $\operatorname{Im}\phi$.

Let $e_{i,p} = \phi(E_{p,1}[l_i \times k_i])$. For any $b \in E$,

$$e_{i,p}(b)e^{i,p} = bE_{p,1}[l_i \times k_i]E_{1,p}[k_i \times l_i] = bE_{p,p}[l_i \times l_i],$$

so $\sum_{p=1}^{l_i} e_{i,p}(b)e^{i,p} = \sum_{p=1}^{l_i} bE_{p,p}[l_i \times l_i] = bI_{l_i}$,

hence $\sum_{i=1}^{N}\sum_{p=1}^{l_i} e_{i,p}(b)e^{i,p} = \sum_{i=1}^{N} bI_{l_i} = b\sum_{i=1}^{N} I_{l_i} = b1_A = b$

Thus for any $b \in E$, $b = \sum_{i=1}^{N}\sum_{p=1}^{l_i} e_{i,p}(b)e^{i,p}$, so that the $e^{i,p}$ and $e_{i,p}$ are the frame and coframe described in Theorem 2.1.14. By Remark 2.1.15, for any $\alpha \in E^\circ$, $\alpha = \sum_{i=1}^{N}\sum_{p=1}^{l_i} e_{i,p}\alpha(e^{i,p})$. Therefore E° is spanned by the set of $e_{i,p}$ which are contained in $\mathrm{Im}\phi$.

This isomorphism is enough to prove that $E \otimes_A E^\circ \simeq A \simeq E^\circ \otimes_A E$, but to prove that E is a line module we need to check that ev and coev in particular are isomorphisms.

Recall that $\mathrm{ev} : E \otimes_A E^\circ \to A$, given by $\mathrm{ev}(b \otimes_A \alpha) = \alpha(b)$.

For any $i \in \{1, ..., N\}$, $q \in \{1, ..., k_i\}$,

$$\begin{aligned}\mathrm{ev}(E_{q,1}[k_i \times l_i] \otimes_A \phi(E_{1,q}[l_i \times k_i])) &= (\phi(E_{1,q}[l_i \times k_i]))(E_{q,1}[k_i \times l_i]) \\ &= E_{q,1}[k_i \times l_i]E_{1,q}[l_i \times k_i] \\ &= E_{q,q}[k_i \times k_i],\end{aligned}$$

so that $\sum_{q=1}^{k_i} \mathrm{ev}(E_{q,1}[k_i \times l_i] \otimes_A \phi(E_{1,q}[l_i \times k_i])) = \sum_{q=1}^{k_i} E_{q,q}[k_i \times k_i] = I_{k_i}$,

hence $\sum_{i=1}^{N}\sum_{q=1}^{k_i} \mathrm{ev}(E_{q,1}[k_i \times l_i] \otimes_A \phi(E_{1,q}[l_i \times k_i])) = \sum_{i=1}^{N} I_{k_i} = 1_A$,

hence $\mathrm{ev}(\sum_{i=1}^{N}\sum_{q=1}^{k_i} E_{q,1}[k_i \times l_i] \otimes_A \phi(E_{1,q}[l_i \times k_i])) = 1_A$. Therefore ev is surjective by A-linearity.

Recall that $\mathrm{coev} : A \to E^\circ \otimes_A E$, given by $\mathrm{coev}(1_A) = \sum_{i=1}^{N}\sum_{p=1}^{l_i} e_{i,p} \otimes_A e^{i,p}$

Let a be any nonzero element of A, let $d_i \in M_{l_i \times l_i}(\mathbb{C})$ be some nonzero component of a, for each $p \in \{1, ..., l_i\}$, $e^{i,p}a = e^{i,p}d_i$ which resides in the $M_{k_i \times l_i}(\mathbb{C})$ component of E and has a first row equal to the pth row of d_i and zero in all other rows. Because d_i is a nonzero matrix there must be some row p that has at least one nonzero entry and therefore an $e^{i,p}$ such that $e^{i,p}a \neq 0$. This means that for any $0 \neq a \in A$, there exists and $e \in E$ such that $ea \neq 0$. By Lemma 3.1.5, this means that coev is injective.

Recall $h : M_{k_i \times k_q}(\mathbb{C}) \to (M_{k_i \times k_j}(\mathbb{C})) \otimes_A (M_{k_j \times k_q}(\mathbb{C}))$ defined by $h(E_{a,b}[k_i \times k_q]) = E_{a,1}[k_i \times k_j] \otimes_A E_{1,b}[k_j \times k_q]$. The map h was proven to be the inverse of $\tilde{f}$ and therefore is an A-bimodule isomorphism.

Thinking in terms of mapping components of A to components of $F \otimes_A E$, let

$$h_i : M_{l_i \times l_i}(\mathbb{C}) \to (M_{l_i \times k_i}(\mathbb{C})) \otimes_A (M_{k_i \times l_i}(\mathbb{C}))$$

be defined by $h_i(E_{a,b}[l_i \times l_i]) = E_{a,1}[l_i \times k_i] \otimes_A E_{1,b}[k_i \times l_i]$. Then

$$\begin{aligned} h_i(I_{l_i}) &= h_i(\sum_{p=1}^{l_i} E_{p,p}[l_i \times l_i]) \\ &= \sum_{p=1}^{l_i} h_i(E_{p,p}[l_i \times l_i]) \\ &= \sum_{p=1}^{l_i} E_{p,1}[l_i \times k_i] \otimes_A E_{1,p}[k_i \times l_i] \\ &= \sum_{p=1}^{l_i} \phi^{-1}(e_{i,p}) \otimes_A e^{i,p}. \end{aligned}$$

On each component h_i is an A-bimodule isomorphism, so the direct sum $h = \oplus_{i=1}^{N} h_i :$ $A \to F \otimes_A E$ is an A-bimodule isomorphism. Hence

$$\begin{aligned} h(1_A) &= h(\sum_{i=1}^{N} I_{l_i}) \\ &= \sum_{i=1}^{N} h(I_{l_i}) \\ &= \sum_{i=1}^{N} h_i(I_{l_i}) \\ &= \sum_{i=1}^{N} \sum_{p=1}^{l_i} \phi^{-1}(e_{i,p}) \otimes_A e^{i,p}. \end{aligned}$$

The map $\phi \otimes_A \mathrm{id}_E : F \otimes_A E \to E^\circ \otimes_A E$ is a tensor product of two isomorphisms and so is an isomorphism itself, this gives us the composition of isomorphisms

$(\phi \otimes_A \mathrm{id}_E) \circ h : A \to E^\circ \otimes_A E$. At last,

$$\begin{aligned}
((\phi \otimes_A \mathrm{id}_E) \circ h)(1_A) &= (\phi \otimes_A \mathrm{id}_E)(h(1_A)) \\
&= (\phi \otimes_A \mathrm{id}_E)(\sum_{i=1}^{N}\sum_{p=1}^{l_i} \phi^{-1}(e_{i,p}) \otimes_A e^{i,p}) \\
&= \sum_{i=1}^{N}\sum_{p=1}^{l_i} (\phi \otimes_A \mathrm{id}_E)(\phi^{-1}(e_{i,p}) \otimes_A e^{i,p}) \\
&= \sum_{i=1}^{N}\sum_{p=1}^{l_i} \phi(\phi^{-1}(e_{i,p})) \otimes_A \mathrm{id}_E(e^{i,p}) \\
&= \sum_{i=1}^{N}\sum_{p=1}^{l_i} e_{i,p} \otimes_A e^{i,p}
\end{aligned}$$

This means that $(\phi \otimes_A \mathrm{id}_E) \circ h = \mathrm{coev}$, but we know that $(\phi \otimes_A \mathrm{id}_E) \circ h$ is an isomorphism, so it follows that coev is an isomorphism.

As already proven, if coev is an isomorphism and ev is surjective it follows that ev is injective.

Therefore E is a line module. □

4.4 Conclusions

Putting Theorems 4.1.5, 4.2.1, and 4.3.1 together gives us our main conclusion for this chapter.

Corollary 4.4.1. *For a semisimple complex algebra $A \simeq \oplus_{i=1}^{N} M_{k_i}(\mathbb{C})$, $Pic(A) \simeq S_N$.*

We now apply this to the special case of group rings.

Proposition 4.4.2. *Let G be a finite group, and $\mathbb{C}G = \{\sum_{g\in G} c_g g | c_g \in \mathbb{C}\}$ the* group algebra *of G over $\mathbb{C}$. Then $Pic(\mathbb{C}G) \simeq S_r$ where r is the number of conjugacy classes of G.*

Proof. By Section 18.2 Theorem 10 of Dummit and Foote [DF04]:

$\mathbb{C}G \simeq M_{n_1}(\mathbb{C}) \oplus ... \oplus M_{n_r}(\mathbb{C})$, where r is equal to the number of conjugacy classes of G.

Let $A = M_{n_1}(\mathbb{C}) \oplus ... \oplus M_{n_r}(\mathbb{C})$.

For any A-bimodule E, $E \simeq \oplus_{i,j=1}^{r} M_{k_i \times k_j}(\mathbb{C})^{m_{i,j}}$.

The multiplicity matrix of an A-bimodule is an $r \times r$ matrix and

$\text{Pic}(A) = \text{Inv}(\text{Bimod}(A)) \simeq \text{Inv}(M_r(\mathbb{Z}_{\geq 0})) \simeq S_r$. $\square$

Example 4.4.3. Let S_3 be the permutation group on 3 elements. The group algebra of S_3 is $\mathbb{C}S_3 = \{\sum_{s \in S_3} c_s s | c_s \in \mathbb{C}\}$. The group S_3 has 3 conjugacy classes, and $\mathbb{C}S_3 \simeq M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$ (from Dummit and Foote, Section 18.2, Example 2 [DF04]). Let $A = M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$.

For any A-bimodule E, $E \simeq \oplus_{i,j=1}^{3} M_{k_i \times k_j}(\mathbb{C})^{m_{i,j}}$.

$\oplus_{i,j=1}^{3} M_{k_i \times k_j}(\mathbb{C})^{m_{i,j}} = M_{1\times 1}(\mathbb{C})^{m_{1,1}} \oplus M_{1\times 1}(\mathbb{C})^{m_{1,2}} \oplus M_{1\times 2}(\mathbb{C})^{m_{1,3}} \oplus M_{1\times 1}(\mathbb{C})^{m_{2,1}} \oplus M_{1\times 1}(\mathbb{C})^{m_{2,2}} \oplus M_{1\times 2}(\mathbb{C})^{m_{2,3}} \oplus M_{2\times 1}(\mathbb{C})^{m_{3,1}} \oplus M_{2\times 1}(\mathbb{C})^{m_{3,2}} \oplus M_{2\times 2}(\mathbb{C})^{m_{3,3}}$

The multiplicity matrix of an A-bimodule is a 3×3 matrix and

$\text{Pic}(A) = \text{Inv}(\text{Bimod}(A)) \simeq \text{Inv}(M_3(\mathbb{Z}_{\geq 0})) \simeq S_3$.

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