A HYPERFACTORIZATION OF ORDER 8, INDEX 2

by

J.D. Horton

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J.D. Horton

School of Computer Science

University of New Brunswick

Consider the complete graph on 2n vertices, K_{2n}. A one-factor of K_{2n} is a set of n edges (pairs of vertices) such that each vertex is in exactly one of the edges. A hyperfactorization of index λ, or λ-hyperfactorization, is a family of one-factors such that each pair of edges without vertices in common occur together in exactly λ of the one-factors. A hyperfactorization is said to be simple if all the one-factors are distinct.

A trivial hyperfactorization can be constructed by taking the family of all one-factors. Each pair of edges occur together in (2n-5)(2n-7)...(1) one-factors, the number of one factors in K_{2n-4}. In particular for K_8, n=4, the trivial hyperfactorization has index 3.

The only known non-trivial hyperfactorizations of index 1 are due to Cameron (1976), where the orders are of the form $2n = 2^a + 2$, $a \geq 3$. Boros, Jungnickel and Vanstone have (1987) constructed recursively non-trivial simple λ-hyperfactorizations of K_{2n} for all $n \geq 5$, but with much larger indexes. Jungnickel and Vanstone (1987) have constructed two other examples, 2n = 12, λ = 15, and 2n = 24, λ = 495. These are the only known constructions. Two non-existence results are that no hyperfactorization of index 1 exists for K_8 (Cameron(1976), Mathon (1981)), nor for K_{12} (Lam et al, 1983).

In this note, we are interested only in K_8 and λ = 2. Clearly, a simple 2-hyperfactorization does not exist, for if one did, the complementary hyperfactorization with respect to the trivial hyperfactorization ($\lambda = 3$) would be a hyperfactorization of index 1. However, a non-simple hyperfactorization of K_{2n} with index 2 can be constructed. Let the vertex set of K_8 be $V = \{\infty\} \quad {}_7$, where ${}_7 = \{0,1,\ldots,6\}$. V can be thought of as the integers modulo 7 extended with ∞, so that $i + \infty = \infty$, $i \times \infty = \infty$. Thus integers can act on a one-factor of K_8 using either addition (modulo 7) or multiplication (modulo 7) (but not by zero) and obtain another one-factor.

A 2-hyperfactorization of K_8 must consist of $7 \times 5 \times 2 = 70$ one-factors. We assume that the hyperfactorization admits an automorphism of order 7, and can be generated by 10 "base" one-factors. The other one-factors are produced by adding (modulo 7) 1,2,3,4,5,6 to the base one-factors. Each of the base one-factors can also be assumed to include the edge $\{\infty\ 0\}$.

Consider the following one-factors (for simplicity the edge $\{i,j\}$ is written ij):

$$A = \{0\infty,\ 16,\ 25,\ 34\}$$

$$B = \{0\infty,\ 15,\ 23,\ 46\}$$

$$C = \{0\infty,\ 12,\ 34,\ 56\}$$

$$D = \{0\infty,\ 12,\ 36,\ 45\}$$

The following 10 one-factors form a set of base one-factors for a 2-hyperfactorization of K_8: A, two copies of B, $6 \times$ B, C, $2 \times$ C, $3 \times$ C, D, $2 \times$ D, $4 \times$ D. It is tedious but not difficult to verify that all edges occur together twice in the 70 one-factors generated by these 10 base one-factors.

We say that the edge ij has <u>length</u> $m = |i-j|$, where $m = 1,2$, or 3, and that it has <u>sum</u> $s = i+j$ (modulo 7). The length and sum of an edge ij determine what the edge is ($i = (s-m)|2$, $j = (s+m)|2$).

All pairs involving edges with an ∞ occur together exactly twice if the base set contains all edges, not including ∞, exactly twice. Alternatively, for each length m, edges of all possible sums, excluding $s = \pm m$, must occur twice. The sums $\pm m$ are excluded because the edges of length m with these sums contain 0, which conflicts with the 0∞ edge.

All pairs involving edges of the same length m must occur together twice. The difference between the sums of the edges cannot be $\pm m$, since then the edge would be of the form $(i, i+m)$ $(i+m, i+2m)$ which cannot be together in a one-factor because they have a vertex in common. The other two possible differences from 1,2,3 must occur exactly twice, as the difference between the sums of edges of length m in the same base one-factor.

For edges of different length, say k and m, $k \neq m$, there are four differences between the sums of edges that cannot occur, $\pm(k\pm m)$. The other three differences from 0,1,2,3,4,5,6 must occur twice between sums of edges, one of length m and the other of length k, both in the same base one-factor. This is necessary and sufficient to make every pair of edges, of different length and without ∞, to occur together twice in the 70 one-factors generated by the base one-factors.

These ten base one-factors satisfy the above three conditions, as is shown in Table 1. Therefore, they generate a 2-hyperfactorization of K_8.

A computer search was carried out to find similar sets of base one-factors for K_{14} and K_{16} with $\lambda = 1$, but none were found.

Base One-Factors	Lengths	Sums of edges 1	2	3	Differences between edges: Same length 1	2	3	Different lengths 1-2	1-3	2-3
A	= $\{0\infty,16,25,34\}$	0	0	0	-	-	-	0	0	0
B (twice)	= $\{0\infty,15,23,46\}$	5	3	6	-	-	-	2	6	4
6×B	= $\{0\infty,62,54,31\}$	2	4	1	-	-	-	5	1	3
C	= $\{0\infty,12,34,56\}$	034	-	-	133	-	-	-	-	-
2×C	= $\{0\infty,24,61,35\}$	-	016	-	-	112	-	-	-	-
3×C	= $\{0\infty,36,25,14\}$	-	-	025	-	-	223	-	-	-
D	= $\{0\infty,12,36,45\}$	23	-	2	1	-	-	-	01	-
2×D	= $\{0\infty,24,65,13\}$	4	46	-	-	2	-	05	-	-
4×D	= $\{0\infty,41,53,26\}$	-	1	15	-	-	3	-	-	03
Forbidden values		1	25	34	2	3	1	1346	2345	1256

Table 1 - Sums and Differences

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References

E. Boros, D. Jungnickel and S.A. Vanstone, The existence of non-trivial hyperfactorizations of K_{2n}, preprint (1987).

P.J. Cameron, Parallelisms of complete designs, London Mathematical Lecture Notes 23, Cambridge University Press, Cambridge (1976).

D. Jungnickel and S.A. Vanstone, Hyperfactorizations of graphs and 5-designs, J. Univ. Kuwait (Sci.) 14 (1987), 213-224.

C.W.H. Lam, L. Thiel, S. Swiercz and J. McKay, The nonexistence of ovals in a projective plane of order 10, Discrete Mathematics 45 (1983), 318-321.

R. Mathon, The partial geometries (5,7,3), Congressus Numeratium 31 (1981), pp. 129-139.