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Abstract.

An adaptation of a method by Boland [3] leading to a generalization of the block methods of Linz [8] are presented in this paper. Conditions for such methods to be convergent and A-stable are given along with some numerical examples.

1. Introduction.

Consider the equation

$$(1) \qquad y(x) = g(x) + \int_0^x K(x,t,y(t))dt \qquad 0 \leq x \leq a$$

Assume that $g(x) \varepsilon C[0,a]$, that $K(x,t,u)$ satisfies a uniform Lipschitz condition with respect to u and that $K(x,t,u)$ is uniformly continuous in x and t for all $(x,t,u) \varepsilon R$ where

$$R = \{(x,t,u): \quad 0 \leq t \leq x \leq a, \ |u| < \infty \}$$

It is shown in [10] that under these conditions (1) has a unique solution $y(x) \ \varepsilon \ C[0,a]$.

Volterra integral equations have appeared in the literature in connection with such things as developing mathematical models for studying the spread of infectious diseases [9].

Block methods of solution were first suggested by Young [11] and more recently Linz [8] extended the idea. The advantages of the methods, as Linz indicated, are that no starting values are required and changing the step size is possible after each block of values

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is determined. Although we shall deal with nonsingular problems, integral equations with singularities have been solved using step by step and block methods in [5,6,7].

To recall the form of block methods, let I(h) denote a partition of [0,a] where

$$I(h) = \{x_j : x_j = jh,\ j = 0,1,\ldots,\ N, h>0,\ Nh = a\}$$

Let

$$(2)\quad y_n = g(x_n)+h\sum_{i=0}^{mp} W_{ni}K(x_n,x_i,y_i)+h\sum_{i=mp+1}^{(m+1)p} W'_{ki}K(x_n,x_i,y_i)$$

$$n = mp+k,\ K = 1,2,\ldots,p$$

be a system of p equations where y_i denotes an approximation to $y(x_i)$, p is an integer and m is the integer part of n/p. The weights W_{ni} and W'_{ki} depend on the quadrature rules used.

In this paper we shall present a generalization of methods of the form (2) and establish conditions for convergence. In addition, we shall provide conditions for which such methods are A-stable.

2. Quadrature Coefficients.

Let $f = f(x,y(x))$ be a function defined for all $x \in [0,a]$. For integers p and q, let

$$J = \{ u_i : 0 \leq u_0 \leq u_1 \leq \ \ldots\ u_q \leq x_p\}$$

Then, for each k, k=1,2,...,p, define a polynomial approximation $g_k(x,y)$ to $f(x,y)$ by

$$g_k(x,y) = \sum_{i=0}^{p}\sum_{j=0}^{q} \ell_i(x)\, L_j(x)\, f(x_j,y_i)$$

where

$$\ell_i(x) = Q(x) / (x-x_i)\, Q'(x_i) \quad , \quad Q(x) = \prod_{i=0}^{p} (x-x_i)$$

and

$$L_j(x) = P(x)/(x-u_j)\, P'(u_j) \quad , \quad P(x) = \prod_{j=0}^{q} (x-u_j)$$

In particular, weights $\{W'_{kij}\}$ can now be defined by

$$W'_{kij} = (1/h) \int_0^{x_k} \ell_i(x)\, L_j(x)dx$$

Using these weights, a generalization of (2) is given by

$$(3) \quad y_n = g(x_n) + h \sum_{\ell=0}^{m-1} \sum_{i=0}^{p} \sum_{j=0}^{q} W_{nij}\, K(x_n, x_{p\ell,j}, y_{i+p\ell})$$

$$+ h \sum_{i=0}^{p} \sum_{j=0}^{q} W'_{kij}\, K(x_n, x_{mp,j}, y_{i+mp})$$

$$n = mp+k,\; k = 1,2,\dots,p.$$

The weights $\{W_{nij}\}$ are those of the quadrature formula defined by the weights $\{W'_{pij}\}$ used compositely and $x_{p\ell,j} = x_{p\ell} + u_j$

The above method is an adaptation from those used in solving linear Fredholm equations [3] and in addition to the previously mentioned advantages of block methods, it permits greater freedom in representing the kernel at a number of points.

3. Theoretical Results.

Definition: A quadrature formula is said to be convergent if for any continuous function $f(x,y(x))$ defined for all $X \in [0,a]$,

$$\left|\int_0^{x_s} f(x,y(x))dx - h \sum_{\ell=0}^{m} \sum_{i=0}^{p} \sum_{j=0}^{q} W_{nij} f(x_{p\ell,j}, y(x_{i+p\ell}))\right| \to 0$$

as $s \to \infty$, $h \to 0$ such that $x_s \in I(h)$ and m is the integer part of s/p.

With this definition, we can use the definitions of order and convergence that are generally applied ([4], [8]). In addition, we shall need the following lemmas to establish that methods of the form (3) are convergent and have a specified order.

Lemma 1. Let $\{V_i\}$ be a sequence of numbers satisfying

$$|V_n| \leq A \sum_{i=0}^{n-1} |V_i| + B \qquad A,B > 0$$

then $|V_n| \leq B(1+A)^n \qquad n = 0,1,\ldots,N$

In particular, if $A = hM$, $M > 0$ and $x = nh$, then

$$|V_n| \leq Be^{Mx} \qquad n = 0,1,\ldots,N$$

Proof: see [8]

Lemma 2. Let $f(x,y)$ be a function which is continuously differentiable on $[a,b]$ $q+1$ times with respect to x and $p+1$ times with respect to y. Let $x_0, x_1, \ldots, x_q$ and $t_0, t_1 \ldots, t_p$ be partitions of $[a,b]$. Let $\ell_i(x)$ and $L_j(x)$ be Lagrange polynomials and $y_i = y(t_i)$. Then the function

$$g(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{q} \ell_i(x) L_j(x) f(x_j, y_i)$$

is an approximating function for $f(x,y)$ with an error term given by

$$E(f) = \frac{W_q(x)}{(q+1)!}\frac{\partial^{q+1} f(\xi,y)}{\partial x^{q+1}} + \frac{V_p(x)}{(p+1)!}\frac{\partial^{p+1} f(x,\eta)}{\partial y^{p+1}}$$

$$- \frac{W_q(x)V_p(x)}{(q+1)!(p+1)!}\frac{\partial^{q+p+2} f(\xi',\eta')}{\partial x^{q+1}\,\partial y^{p+1}}$$

where $\eta = y(\xi_1)$, $\eta' = y(\xi_2)$, $\xi,\xi_1,\xi_2,\xi' \in (a,b)$, $W_q(x) = \prod_{j=0}^{q}(x-x_j)$

and $V_p(x) = \prod_{i=0}^{p}(x-t_i)$

Proof: The proof is analogous to that for finding the error in an interpolating polynomial for a function of one variable. In particular, the integrated form of the error for the special case $f(x,y(x)) = g(x)\,y(x)$ is given in [2].

If the partitions in lemma 2 are chosen to be equally spaced with $x_j = a + jh_1$, and $t_i = a + ih_2$, then the error term can be rewritten as

$$(4)\quad E(f) = \frac{h_1^{q+1}}{(q+1)!}\frac{\partial^{q+1} f(\xi,y)}{\partial x^{q+1}}\prod_{j=0}^{q}(u_1-j) + \frac{h_2^{p+1}}{(p+1)!}\frac{\partial^{p+1} f(x,\eta)}{\partial y^{p+1}}\prod_{i=0}^{p}(u_2-i)$$

$$\frac{-h_1^{q+1}\,h_2^{p+1}\,\partial^{p+q+2} f(\xi',\eta')}{(q+1)!\,(p+1)!\,\partial x^{q+1}\,\partial y^{p+1}}\prod_{j=0}^{q}(u_1-j)\prod_{i=0}^{p}(u_2-i)$$

where $u_1 h_1 = x-x_0$ and $u_2 h_2 = x-t_0$

Returning to equation (3), we first obtain a criteria for convergence in the case of iterating to obtain approximations for implicit systems. Let the iterates for y_n be denoted by $y_n^{(v)}$, $v=0,1...$ where $y_n^{(0)}$ is an initial predicted value. Subtracting

$$y_n^{(v)} = g(x_n) + h \sum_{\ell=0}^{m-1} \sum_{i=0}^{p} \sum_{j=0}^{q} W_{nij}\, K(x_n, x_{p\ell,j}, y_{i+p\ell})$$

$$+h \sum_{j=0}^{q} W_{kij}\, K(x_n, x_{mp,j}, y_{mp}) + h \sum_{i=0}^{p} \sum_{j=0}^{q} W'_{kij} K(x_n, x_{mp,j}, y_{i+mp}^{(v-1)}$$

from equation (3), it follows using standard arguments that $y_n^{(v)} \to y_n$ as $v \to \infty$ provided $hAL < 1$ where L is the Lipschitz constant satisfied by $K(x,t,u)$ with respect to u and

$$A = p\underset{1 \leq i,\ k \leq p}{\mathrm{MAX}} \sum_{j=0}^{q} |W'_{kij}|$$

Conditions for methods of the form (3) to be convergent are given in the following theorem.

Theorem 1. Assume

(a) $|K(x,t,u) - K(x,t,u^*)| \leq L\, |u-u^*|$

for all (x,t,u) and $(x,t,u^*) \varepsilon R$

(b) $W = p \text{ Max } \{ 2 \sum_{j=0}^{q} | W_{nij}|, \sum_{j=0}^{q} |W'_{kij}|\}$

(c) $R_n(h) \to 0$ as $h \to 0$ for $n = mp+k$, $K = 1,2,..,p$ where

$$y(x_n) = g(x_n) + h \sum_{\ell=0}^{m-1} \sum_{i=0}^{p} \sum_{j=0}^{q} W_{nij}\, K(x_n, x_{p\ell,j}, y(x_{i+p\ell}))$$

$$+ h \sum_{i=0}^{p} \sum_{j=0}^{q} W'_{kij}\, K(x_n, x_{mp,j}, y(x_{i+mp})) + R_n(h)$$

Then block methods of the form (3) are convergent

Proof: The proof is analogous to that given in [8] for methods of the form (2). Therefore only a brief outline is given.

Let $x=x_n$ in equation (1). Subtract equation (1) from equation (3), subtract the equation in (c) from equation (1) and take the absolute value of the sum. The resulting system can be written

$$||e_m|| \leq hWL \sum_{i=0}^{m-1} ||e_i|| + hWL \cdot ||e_m|| + R_m^*(h)$$

where

$$||e_r|| = \underset{1\leq k\leq p}{MAX} \{| y_{rp+k} - y(x_{rp+k})|\} \text{ and } R_r^*(h) = \underset{1\leq k\leq p}{MAX} \{ | R_{rp+k}(h)|\}$$

Solving for $||e_m||$ and using lemma 1, it follows that the methods are convergent.

Working with the integrated form of equation (4), a sharper error bound can be obtained. Suppose, for example, $u_i=x_i$, $i=0,1,2,\ldots,p$. Then equation (4) simplifies as $h_1 = h_2 = h$. In addition assume that the quadrature formula is closed with p even. For this case, we can add the following corollary.

Corollary 1. Assume $K(x,t,u)$ has continuous partial derivatives of order $p+2$ with respect to t and u. Then

$$R_n(h) = \Big\{ \frac{1}{(p+2)!} \frac{\partial^{p+2} K(x,\xi_1,\eta)}{\partial t^{p+2}} + \frac{1}{(p+1)!} \frac{\partial^{p+2} K(x,\xi_2,\eta)}{\partial t^{p+1}\,\partial u} + \frac{m}{(p+2)!} \frac{\partial^{p+2} K(x,\xi,\eta_1)}{\partial u^{p+2}} + \frac{1}{(p+1)!} \frac{\partial^{p+2} K(x,\xi,\eta_2)}{\partial t\,\partial u^{p=1}} \Big\} h^{p+3} \int_0^p t\Pi_p(t)dt$$

$$- \frac{m}{[(p+1)!]^2} \frac{\partial^{2p+2} K(x,\xi_3,\eta_3)\, h^{p+3}}{\partial t^{p+1}\,\partial u^{p+1}} \int_0^p \Pi_p(t)^2\, dt$$

where $\Pi_p(t) = t(t-1)\ldots(t-p)$ and m is the number of times the formula is used compositely.

For the case, p odd, a similar result can be stated (see, [2]). Using these, a sharper bound for $R_m^*(h)$ in the theorem can be obtained depending on the quadrature formulae used.

4 A-Stability.

Definition. An approximating method is said to be A-stable if when applied to the problem

$$(5) \qquad y(x) = 1 + \lambda \int_0^x y(t)dt \qquad \lambda<0$$

with arbitrary step h, then

$$\lim_{\substack{i \to \infty \\ h \text{ fixed}}} y_i = 0$$

Solving (5) by a method of the form (3), we get

$$(6) \quad y_n = 1 + \lambda h \sum_{i=0}^{mp} W_{ni} y_i + \lambda h \sum_{i=mp+1}^{(m+1)p} W'_{ki} y_i$$

$$n = mp+k \;,\; K = 1,2,\dots,p$$

Subtracting each equation in (6) from

$$y_{mp} = 1 + \lambda h \sum_{i=0}^{mp-1} W_{ni} y_i + h\lambda d_0 y_{mp} \qquad (d_0,\ \text{constant})$$

and expressing the resulting system in matrix form yields

$$(7) \quad (I_p - \lambda hB)\ Y_m = (E + \lambda hD)\ y_{mp}$$

Here, I_p is the pxp identity matrix, $Y_m = (y_{mp+1},\dots,y_{mp+p})^T$, $E = (1,1,\dots,1)^T$, $D = (d_1,d_2,\dots d_p)^T$, ($d_i$, constant) and B is the matrix $\{W'_{ki}\}$.

Without loss of generality, set h=1. Let $A^{(0)}(\lambda) = I_p - \lambda B$ and define $A^{(i)}(\lambda)$ i=1,2,...,p to be the matrix obtained by replacing the i^{th} column of $A^{(0)}(\lambda)$ by the vector $E+\lambda D$. Define the polynomials $P_k(\lambda)$ of degree $\leq p$ by

$$P_k(\lambda) = \det (A^{(k)}(\lambda)) \qquad K=0,1,\ldots,p$$

We note that $P_0(0) = P_p(0) = 1$. The following result was established in [12] and applies as well to block methods of the form (3).

Theorem 2. Let $\lambda<0$. A necessary and sufficient condition for a block method of the form (3) to be A-stable is that $| P_p(\lambda) / P_0(\lambda) | < 1$.

The problem then is to construct such polynomials. Recalling equation (7), the elements of B are the weights $\{W'_{ki}\}$ where

$$W'_{ki} = \sum_{j=0}^{q} W'_{kij} = (1/h) \int_0^{x_k} \ell_i(x) \sum_{j=0}^{q} L_j(x)dx = (1/h)\int_0^{x_k}\ell_i(x)dx$$

Thus the same matrix B results regardless of the choice of the interpolating polynomial $L_j(x)$. Therefore, it is the polynomials $\ell_i(x)$ which determine the A-stability of the method. A discussion regarding the construction of such polynomials is given in [12].

5. Numerical Examples.

To define a particular fourth order method, let p=q=2. Then

$$\{W'_{1ij}\} = \frac{1}{120}\begin{bmatrix} 31 & 23 & -4 \\ 23 & 64 & -7 \\ -4 & -7 & 1 \end{bmatrix}$$

$$\{W'_{2ij}\} = \frac{1}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

where $W'_{kij} = (1/h) \int_0^{x_k} \ell_i(x) \ell_j(x)dx \qquad k=1,2$
and $\ell_i(x)$ is the Lagrange polynomial interpolating at the nodes x_0, x_1 and x_2.

An example of an A-stable method results if we set

$$\{W'_{1ij}\} = \frac{1}{120} \begin{bmatrix} 32 & 16 & -8 \\ 21 & 78 & 1 \\ -3 & -14 & -3 \end{bmatrix} .$$

and $\{W'_{2ij}\}$ as in the first method. The weights $\{W'_{1ij}\}$ were obtained using the methods in [1] extended to the case of two dimensions.

We shall refer to these methods respectively as Method 1 and Method 2. The following examples were solved using thses two methods.

Example 1. $y(x) = 3 + 2x - \int_0^x (2(x-t)+3)\ y(t)\ dt$

Exact solution $y(x) = 4\exp(-2x) - \exp(-x)$

Example 2. $y(x) = \exp(x^2) - x + x\exp(x^2/2) - \int_0^x xt\sqrt{y(t)}\ dt$

Exact solution $y(x) = \exp(x^2)$

Example 2 requires that we iterate to obtain solutions. The iteration process is stopped as soon as $||e_m|| \leq 10^{-8}$. Table 1 is a table of absolute errors for the various step sizes.

We note that example 1 appeared in [8] . The numerical results for method 1 are similar to those given there even though the corresponding modification has not been extended to the present methods. This can be done in a straight forward manner and so the details have not been repeated here.

TABLE 1

x	h=.05		h=.10		h=.20	
			Example 1			
	Method 1	Method 2	Method 1	Method 2	Method 1	Method 2
1.	3.9×10^{-6}	8.4×10^{-5}	6.2×10^{-5}	6.2×10^{-4}	1.9×10^{-4}	1.8×10^{-3}
2.	7.1×10^{-7}	4.9×10^{-6}	1.1×10^{-5}	3.3×10^{-5}	1.9×10^{-4}	1.9×10^{-4}
3.	3.4×10^{-10}	6.4×10^{-6}	2.1×10^{-8}	4.9×10^{-5}	5.4×10^{-5}	3.5×10^{-4}
4.	5.6×10^{-8}	3.9×10^{-6}	9.0×10^{-7}	3.0×10^{-5}	1.5×10^{-5}	2.2×10^{-4}
			Example 2			
1.	5.0×10^{-7}	1.9×10^{-6}	7.9×10^{-6}	1.9×10^{-5}	3.0×10^{-4}	2.6×10^{-3}
2.	2.6×10^{-5}	1.2×10^{-4}	4.1×10^{-4}	1.2×10^{-3}	6.2×10^{-3}	1.2×10^{-2}
3.	1.4×10^{-3}	2.8×10^{-3}	2.2×10^{-2}	3.3×10^{-2}	1.6	7.3
4.	1.5×10^{-1}	1.6×10^{-1}	2.3	2.4	3.3×10^{1}	3.5×10^{1}

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