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CANADA

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Abstract

A resolvable (balanced) path design, RBPD (v,k,) is the decomposition of λ copies of the complete graph on v vertices into edge-disjoint subgraphs such that each subgraph consists of v/k vertex-disjoint paths of length k-1 (k vertices). It is shown that an RBPD $(v,3,\lambda)$ exists if and only if $v \equiv 9$ (modulo $12/\gcd(4,\lambda)$). Moreover, the RBPD $(v,3,\lambda)$ can have an automorphism of order v/3. For $k > 3$, it is shown that if v is large enough, then an RBPD (v,k,1) exists if and only if $v \equiv k^2$ (modulo lcm(2k-2,k)). Also, it is shown that the categorical product of a k-factorable graph and a regular graph is also k-factorable. These results are stronger than two conjectures of Hell and Rosa.

1. INTRODUCTION

P. Hell and A. Rosa in [4] studied the effect of products upon the decomposition of graphs into various subgraphs. In particular, they studied balanced path designs, the decomposition of the complete graph into paths of a given length, so that each vertex is in an equal number of paths. Two conjectures from that paper are proved in this paper: (1) If two graphs have n-factorizations, then so does their categorical product; (2) the complete graph on v vertices can be decomposed into sets of paths of length two such that each vertex appears exactly once in each set of paths or resolution class, if and only if $v \equiv 9 \pmod{12}$. Stronger theorems than either conjecture are proven. In fact, the existence question for resolvable path designs with $\lambda = 1$ is solved asymptotically.

In this paper, graphs are considered to be undirected unless otherwise stated. Edges are identified with pairs of endpoints but multiple edges are allowed. V(G) denotes the vertex set of the graph G, and E(G) denotes the edge set of G. The <u>categorical product</u> of two graphs G and H (both undirected, or both directed), $G \otimes H$, is defined by:

$$V(G \otimes H) = \{(v,u) \mid v \in V(G) \text{ and } u \in V(H)\} ,$$

$$E(G \otimes H) = \{((v,u),(v',u')) \mid (v,v') \in E(G) \text{ and } (u,u') \in E(H)\} .$$

Let G be any graph. If there is a set S of subgraphs of G, such that each edge of G is in precisely one member of S, S is said to be a <u>decomposition</u> of G. An <u>n-factor</u> of a graph G is a spanning subgraph that is regular of degree n, that is, all vertices of G have degree n in the subgraph. A decomposition of G into n-factors is called an n-factorization of G.

The complete graph on v vertices, in which each vertex is joined precisely once to each other vertex, is denoted by K_v. Then λK_v denotes the graph with v vertices in which each vertex is joined precisely λ times to each other vertex. A decomposition of λK_v into graphs isomorphic to a graph G is called a G-design. If G is a path with k vertices, the G-design is called a path design. If each vertex occurs in precisely the same number of subgraphs in the decomposition, the G-design is said to be balanced. Balanced path designs are called handcuffed designs by Lawless [7], [8], and by Hung and Mendelsohn [5], [6]. If the condition that the design be balanced is dropped, then the existence problem was solved by M. Tarsi in [11]. A G-design S is said to be resolvable if S can be partitioned into sets of subgraphs, called resolution classes, such that each vertex occurs precisely once in each resolution class. Note that if a G-design is resolvable then it is automatically balanced. Following Hell and Rosa, I denote a balanced path design by BPD(v,k,λ) where v is the number of vertices of the complete graph, λ is the number of copies of the complete graph (the multiplicity of each edge), and k is the number of vertices in the path P. If the path design is resolvable, denote it by RBPD(v,k,λ).

The following examples show how these definitions relate to better known combinatorial structures. A K_k-decomposition of λK_v is simply a balanced incomplete block design BIBD(v,k,λ). An RBPD(v,2,1) is a one-factorization of K_v.

2. THE PRODUCT THEOREM

Hell and Rosa [4] examine methods to multiply decompositions of graphs. Concerning the categorical product they prove among other results:

(1) If G has a resolvable decomposition into a family F of n-colorable graphs, and if H has a resolvable K_n decomposition, then $G \otimes H$ has a resolvable decomposition into graphs isomorphic to graphs in F.

(2) If G has a resolvable decomposition into a family F of two-colorable graphs, and if H has a resolvable decomposition into edges and circuits, then $G \otimes H$ has a resolvable decomposition into graphs in F.

As corollaries, they obtain for k=1 and k=2 that if G and H both have k-factorizations then $G \otimes H$ has a k-factorization as well. However, they failed to prove the result for $k>2$, and leave the statement as a conjecture ([4], page 237). The following theorem is stronger than the conjecture.

<u>Theorem 1</u>: If G has a k-factorization and H is regular, then $G \otimes H$ has a k-factorization.

<u>Proof</u>: Let H* be the directed graph obtained from H by replacing each edge by two arcs, one directed each way, between the same two vertices. Thus, $V(H^*) = V(H)$ and $E(H^*) = \{(x,y),(y,x) \mid \{x,y\} \in E(H)\}$. Let G' be any directed graph which has G as its underlying undirected graph. Thus $V(G') = V(G)$ and $E(G) = \{\{x,y\} \mid (x,y) \in E(G')\}$. Then $G \otimes H$ is the underlying undirected graph of the directed graph $G' \otimes H^*$. See figure 1.

(Insert figure 1 near here)

Since H is regular, each vertex having degree d say, then each vertex of H* has indegree d and outdegree d. Then H* has a resolvable decomposition into directed circuits, as can be shown by the following standard trick. Split each vertex v into two vertices v_1 and v_2, and let each edge (v,u) be replaced by the edge (v_1,u_2). This new graph which we call H** is bipartite and regular of degree d. As is well-known, such a graph must have a 1-factorization. But a 1-factor of H** corresponds to a 2-factor of H* consisting of directed circuits. Thus H* has a resolvable decomposition into directed circuits.

The categorical product of a k-regular directed graph and a directed circuit is a k-regular graph. Thus the product of a k-factor of G' and a 2-factor of H* consisting of directed circuits is a k-factor of $G' \otimes H^*$. Hence $G' \otimes H^*$, which is the union of such graphs, has a k-factorization. As $G \otimes H$ is the underlying undirected graph of $G' \otimes H^*$, $G \otimes H$ must also have a k-factorization.

<u>Corollary 1</u>: If G and H both have k-factorizations, then so does $G \otimes H$.
<u>Proof</u>: If H has a k-factorization, then H is regular.

The theorem that one would want is "If G and H both have resolvable decompositions into subgraphs isomorphic to some member of a family of graphs $\underset{\sim}{F}$, then so does $G \otimes H$." However, it is not true. Let G be the path of length 2. Then $G \otimes G$ consists of a 4-circuit and a star with 5 vertices (see figure 2). $G \otimes G$ has no resolvable decomposition into paths of length 2. It would be interesting to find other families of graphs, other than k-regular graphs, for which this statement is true.

(insert figure 2 near here)

The only such families that I know with this property are sets of circuits. For example, C_k , $\{$odd circuits$\}$, $\{$even circuits$\}$.

3. RESOLVABLE PATH DESIGNS

If a BPD(v,k,) exists, then it is known that

$$b = \frac{\lambda v(v-1)}{2(k-1)}, \quad r = \frac{\lambda k(v-1)}{2(k-1)}, \quad n = \frac{\lambda (v-1)}{k-1} \tag{1}$$

must all be integers [4]. Here b is the number of paths in the decomposition, r is the number of occurrences of a given vertex in a path, and n is the number of occurrences of a given vertex as the endpoint of a path. Hung and Mendelsohn [6] have shown that these conditions are also sufficient for the existence of a balanced P-design. For the existence of an RBPD(v,k,λ) it is also necessary that k divides v. These facts lead to:

Conjecture 1: An RBPD(v,k,λ) exists if and only if k divides v and that b,r,n as defined by (1) are all integers.

This conjecture can be proven asymptotically for λ=1 using the following two constructions.

Theorem 2: (Hell and Rosa [4, page 241, Corollary 8]). If k is even, and an RBIBD(v,k,λ) exists, then an RBPD(v,k,λ) exists.

The proof starts by noting that an RBPD(k,k,1) exists (it is well-known that K_k decomposes into k/2 hamiltonian paths if k is even). Then each block is decomposed into k/2 paths of length k-1, and hence each resolution class of the block design can be decomposed into k/2 resolution classes of the path design.

Corollary 2: An RBPD(v,4,1) exists if and only if $v \equiv 4$ (modulo 12).

Proof: An RBPD(v,k,1) exists if and only if $v \equiv k^2$ (modulo lcm(2k-2,k)) [4, page 240, Corollary 6]. For k=4, this congruence reduces to $v \equiv 4$ (modulo 12). Also, an RBIBD(v,4,1) exists if and only if $v \equiv 4$ (modulo 12) [3].

But if k is odd, K_k does not decompose into hamiltonian paths. However, a similar theorem is still true, for $\lambda=1$.

Theorem 3: If k is odd, and if an RBIBD(v,k,1) exists with an even number of resolution classes, then an RBPD(v,k,1) exists.

Proof: First, note that K_k can be decomposed into (k-1)/2 paths of length k-1, and one path of length (k-1)/2 [4, page 244, Corollary 11]. Hell and Rosa call such a decomposition a ½ P-decomposition.

Next, the resolution classes of the RBIBD(v,k,1) are arbitrarily paired off. Each pair of resolution classes will be decomposed into k resolution classes of the new path design. For the remainder of the proof, we consider blocks only from one pair of resolution classes.

Consider the blocks to be the vertices of a graph, with two blocks connected by an edge if their intersection is non-empty. Indeed, the intersection can have at most one element since $\lambda = 1$. Thus each edge is associated with one variety of the design. The graph so formed is bipartite with the two resolution classes forming the parts of the bipartition; and is regular of degree k. Hence, this graph has a one-factorization. Let the one-factors be F_1, F_2, F_3,...,F_k.

Each variety of the design is in the intersection of precisely two blocks, one from each resolution class. Thus each variety corresponds to one edge in the graph. Label each variety with the number of the one-factor in which the corresponding edge appears; thus if an edge appears in F_i, label its corresponding vertex i. This procedure labels the varieties with the k numbers 1, 2, ..., k. Moreover, each block contains precisely one variety with any given label.

Now each block can be thought of as a complete subgraph on k varieties, K_k. For each block of one resolution class, decompose the K_k into (k-1)/2 paths of length (k-1)/2. Let the short path be (1, 2, 3,..., (k+1)/2). The (k-1)/2 long paths from each block of the resolution class form (k-1)/2 resolution classes of the path design RBPD(v,k,1).

For the other resolution class we are considering, also decompose each block into (k-1)/2 paths of length k-1 and one path of length (k-1)/2. However, let the latter short path be ((k+1)/2, (k+3)/2,..., k). The long paths again form (k-1)/2 resolution classes of the new path design.

Consider the set of short paths from all the blocks of both resolution classes. Each path joins at the vertex labelled (k+1)/2 with a unique path from the other resolution class to form a path of length k-1. Each vertex of K_v appears precisely once, because precisely v/k vertices of each label-class must occur, one from each block of one of the resolution classes. Thus, this forms a set of vertex-disjoint paths of length k-1 that covers K_v. Thus the pair of resolution classes of the block design has been partitioned into k resolution classes of the path design.

The above proof for k=3 is due to Richard Wilson [12]. For k=3, this proof solved the "handcuffed prisoners" problem totally, since RBIBD(v,3,1) are known to exist for all $v \equiv 3$ (modulo 6) [9], and if $v \equiv 9$ (modulo 12) the number of resolution classes is even.

Corollary 3: (Wilson) An RBPD(v,3,1) exists if and only if $v \equiv 9$ (modulo 12).

For all $k > 3$, the existence of resolvable block designs is known asymptotically. We can, therefore, solve the resolvable path design problem asymptotically.

Theorem 4: Let k be any integer greater than 1. Then there exists a constant c(k) such that if $v > c(k)$, then an RBPD(v,k,1) exists if and only if $v \equiv k^2$ (modulo lcm(2k-2,k)).

Proof: Ray-Chaudhuri and Wilson have proven that for any k there is a constant c(k) such that if $v > c(k)$ and $v \equiv k$ (modulo k(k-1)), then an RBIBD exists [10, page 333, theorem 4]. If an RBPD(v,k,1) exists then

$$v \equiv k^2 \text{ (modulo lcm}(2k-2, k)) \qquad (2)$$

[4, page 240, Corollary 6]. For even k, congruence (2) reduces to $v \equiv k$ (modulo k(k-1)), the same congruence as for resolvable block designs. Theorem 2 completes the proof from even k.

For odd k, congruence (2) reduces to $v \equiv k^2$ (modulo 2k(k-1)). But then $v \equiv k$ (modulo k(k-1)), so that an RBIBD(v,k,1) exists. Moreover, this block design has (v-1)/(k-1) resolution classes. But $(v-1) \equiv k^2-1 \equiv (k+1)(k-1) \equiv 0$ modulo 2(k-1), and hence the number of resolution classes is even. Thus theorem 3 is applicable, and theorem 4 is proven.

4. REGULAR PATH DESIGNS WITH k=3

Conjecture 3 in [4, page 250] is the above conjecture 1 with the constraints that $\lambda=1$, k=3, and the added condition that the design be regular. A P-design is regular if the automorphism group of the design contains a regular automorphism of order v/k. Thus a P-design is regular if it can be constructed using Bose's method of pure and mixed differences (see [2], chapter 15, for example) on the appropriate group. Hell and Rosa use this method in [4] to construct several examples of resolvable P-designs. The construction in the following theorem uses this method for k=3 and $\lambda=1$.

Theorem 5: A regular RBPD(v,3,1) exists if and only if $v \equiv 9 \pmod{12}$.

Proof: The necessity follows from (1) and that 3 divides v. The remainder of the proof consists of constructing a set of paths that generate an RBPD(12t + 9,3,1) using a cyclic group A of order v/3 = 4t + 3. The resolvable P-design is defined on the vertex set
$\{x_i \mid x \in A \text{ and } i = 0,1,2\}$. The subscript defines
the type of the vertex, and is an integer modulo 3.

If x_j and y_i are two vertices, they determine the difference $(x-y)_{ij}$. Note that the differences z_{ij} and $(-z)_{ji}$ are considered to be the same. If i=j, then the difference is said to be pure; if $i \neq j$, then the difference is said to be mixed. We must construct a base set of paths such that each pure and mixed difference occurs precisely once in some path. This property guarantees that all edges occur precisely once when A is applied to the base.

If one set of base paths can be constructed that contains each vertex exactly once, and contains all the pure differences exactly once, then each element of A generates a resolution class with this set of paths. We refer to this set as a base resolution class. The following construction in Table I defines a base resolution class for all t except t=0. Let

$$a = [(t-1)/3] \text{ and}$$

$$b = 4t + 2 - 3a$$

where [x] denotes the greatest integer less than x function. Paths in which the subscript i occurs are to be repeated with the three possible values of i included. The final few blocks are different, depending upon the value of t modulo 3. Pure differences that should be subscripted with an i, and mixed differences that should be subnscripted with i, i+1, are left without subscripts.

Table I. Construction of base resolution class, $k=3, \lambda=1$.

Paths			Differences: Pure	Differences: Mixed
1_i	$(2t)_i$	$(2t+2)_i$	$2t-1 \quad 2$	–
2_i	$(2t-1)_i$	$(2t+3)_i$	$2t-3 \quad 4$	–
3_i	$(2t-2)_i$	$(2t+4)_i$	$2t-5 \quad 6$	–
$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	–
t_i	$(t+1)_i$	$(3t+1)_i$	$1 \quad 2t$	–
0_i	$(2t+1)_i$	$(3t+2)_{i+1}$	$2t+1$	$t+1$
b_i	$(b+2a-1)_{i+1}$	$(b+2a+1)_{i+2}$	–	$2a-1 \quad 2$
$(b+1)_i$	$(b+2a-2)_{i+1}$	$(b+2a+2)_{i+2}$	–	$2a-3 \quad 4$
$(b+2)_i$	$(b+2a-3)_{i+1}$	$(b+2a+3)_{i+2}$	–	$2a-5 \quad 6$
$\vdots$	$\vdots$	$\vdots$		$\vdots \quad \vdots$
$(b+a-1)_i$	$(b+a)_{i+1}$	$(b+3a)_{i+2}$	–	$1 \quad 2a$
Case 1: $t \equiv 1 \pmod 3$, $3a = t-1$, $b = 3t+3$.				
$(b+2a)_0$	$(b+2a)_1$	$(b+2a)_2$	–	$0_{01} \quad 0_{12}$
Case 2: $t \equiv 2 \pmod 3$, $3a = t-2$, $b = 3t+4$.				
$(b+2a)_0$	$(b-1)_1$	$(b+2a)_2$	–	$(2a+1)_{10} \quad (2a+1)_{12}$
$(b-1)_2$	$(b-1)_0$	$(b+2a)_1$	–	$0_{02} \quad (2a+1)_{01}$
Case 3: $t \equiv 0 \pmod 3$, $3a = t-3$, $b = 3t+5$.				
$(b+2a)_i$	$(b-1)_{i+1}$	$(b-2)_{i+2}$	–	$-2a-1 \quad -1$

The remaining differences are all mixed. If these differences can occur in base paths such that one vertex of each type occurs in each path, each path generates a resolution class by itself when A is applied to it. Thus these mixed differences must be paired off so that the two differences have one subscript in common, the other two subscripts being different. Given the differences x_{ij} and y_{jk} the base path chosen is $(0_i \; x_j \; (x+y)_k)$.

Since each base path generates its own resolution class, how the values of x and y are paired off does not matter. Thus the problem has been reduced to examining only the subscripts of the differences, and the number of differences of each type. From another viewpoint, this problem is the same as decomposing K_3 with given edge multiplicities into a set of paths of length 2. But the following lemma is easy to prove:

Lemma: K_3, with edge multiplicities k, ℓ, and m, can be decomposed into paths of length 2 if and only if $k+\ell \geq m$, $k+m \geq \ell$, $m+\ell \geq k$, and $k+\ell+m$ is even.

In the construction of the base resolution class, each mixed difference type occurs the same number of times, except in the final step in cases 1 and 2 where one type may occur once more than some other type. Thus the inequalities stated in the lemma are true. The other condition of the lemma, that the number of mixed differences left is even, can be shown by noting that the total number of differences is even, and that an even number of differences are used in the base resolution class. Therefore, we can find a base set of paths containing each difference, mixed and pure, exactly once such that the pure differences all occur in a base resolution class, and the mixed differences that are not in the base resolution class occur in paths that generate a resolution class. Hence a regular RBPD(v,3,1) exists for all $v \equiv 9 \bmod 12$.

Examples:

(i) $v = 9$, $t = 0$. Does not fit the exact form.

A base resolution class is:

$(1_0\ 0_0\ 0_1)$, $(2_1\ 1_1\ 1_2)$, $(0_2\ 2_2\ 2_0)$.

Resolution class generators are:

$(0_0\ 1_1\ 2_2)$, $(0_2\ 1_0\ 0_1)$, $(0_1\ 2_2\ 1_0)$.

Hell and Rosa had a similar example in [4] for this case.

(ii) $v = 21$, $t = 1$.

A base resolution class is:

$(1_0\ 2_0\ 4_0)$, $(1_1\ 2_1\ 4_1)$, $(1_2\ 2_2\ 4_2)$,
$(0_0\ 3_0\ 5_1)$, $(0_1\ 3_1\ 5_2)$, $(0_2\ 3_2\ 5_0)$,
$(6_0\ 6_1\ 6_2)$.

Resolution class generators are:

$(0_2\ 0_0\ 1_1)$, $(0_1\ 1_2\ 2_0)$, $(0_0\ 3_1\ 6_2)$,
$(0_2\ 3_0\ 0_1)$, $(0_1\ 4_2\ 1_0)$, $(0_0\ 5_1\ 3_2)$,
$(0_2\ 5_0\ 4_1)$, $(0_1\ 6_2\ 5_0)$.

5. THE CASE $k=3, \lambda>1$.

For $\lambda>1$, we can prove:

Theorem 6: An RBPD$(v,3,\lambda)$ exists if and only if, when:

(a) $\lambda \equiv 1$ or 3 (mod 4), then $v \equiv 9$ (mod 12),

(b) $\lambda \equiv 2$ (mod 4), then $v \equiv 3$ (mod 6),

(c) $\lambda \equiv 0$ (mod 4), then $v \equiv 0$ (mod 3).

The resolvable path design can be assumed to be regular.

Proof: The necessity again follows from (1) and that 3 divides v. To complete the proof it is necessary only to construct an RBPD$(v,3,\lambda)$ in the cases:

(a) $\lambda = 2$, $v = 12t + 3$;

(b) $\lambda = 4$, $v = 6t$.

All other cases can be constructed by taking multiple copies of these two cases, or multiple copies of the construction from theorem 5.

Case (a): $\lambda = 2$, $v = 12t + 3$.

The construction is similar to that used in theorem 1, except that each difference must now occur twice. Since one base resolution class does not have enough differences to include all the pure differences, two base resolution classes are needed. Note that A is now the integers modulo $4t+1$. Let $b = 3t + 2$ and $a = [(t-1)/3]$. Table II defines the two base resolution classes.

Table II. Base resolution classes for $v = 12t+3, \lambda=2, t\geq 1$

	Paths			Differences			
				Pure		Mixed	
Both classes:	1_i	$(2t)_i$	$(2t+2)_i$	$2t-1$	2	–	
	2_i	$(2t-1)_i$	$(2t+3)_i$	$2t-3$	4	–	
	⋮	⋮	⋮	⋮	⋮		
	t_i	$(t+1)_i$	$(3t+1)_i$	1	$2t$	–	
	b_i	$(b+2a-1)_{i+1}$	$(b+2a+1)_{i+2}$	–		$2a-1$	2
	$(b+1)_i$	$(b+2a-2)_{i+1}$	$(b+2a+2)_{i+2}$	–		$2a-3$	4
	⋮	⋮	⋮			⋮	⋮
	$(b+a-1)_i$	$(b+a)_{i+1}$	$(b+3a)_{i+2}$	–		1	$2a$
Case 1: $t \equiv 1 \pmod 3$, $b+3a = 4t+1 = 0$:							
Class 1:	$(2t+1)_0$	$(2t+1)_1$	$(b+2a)_2$	–		0_{01}	$(t+2a+1)_{12}$
	$(2t+1)_2$	$(b+2a)_0$	$(b+2a)_1$	–		0_{01}	$(t+2a+1)_{20}$
Class 2:	$(2t+1)_1$	$(2t+1)_2$	$(b+2a)_0$	–		0_{12}	$(t+2a+1)_{20}$
	$(2t+1)_0$	$(b+2a)_1$	$(b+2a)_2$	–		0_{12}	$(t+2a+1)_{01}$
Case 2: $t \equiv 2 \pmod 3$, $b+3a = 4t$:							
Both classes:	0_i	$(2t+1)_{i+1}$	$(b+2a)_{i+2}$	–		$2t+1$	$t+2a+1$
Case 3: $t \equiv 0 \pmod 3$, $b+3a = 4t-1$:							
Both classes:	0_i	$(2t+1)_{i+1}$	$(b+2a)_{i+2}$	–		$2t+1$	$t+2a+1$
Class 1:	$(4t)_0$	$(4t)_1$	$(4t)_2$	–		0_{01}	0_{12}
Class 2:	$(4t)_2$	$(4t)_0$	$(4t)_1$	–		0_{20}	0_{01}

All other differences are included in base paths that generate resolution classes as done in theorem 1. The lemma concerning K_3 can be used again.

Examples:

(iii) $v = 3$, $\lambda = 2$.

The above construction does not apply. However, if the vertices are 0, 1, 2, the paths (0 1 2), (2 0 1), (1 2 0) form an RBPD(3,3,2). This design is a special case of theorem 1, page 234, from [4].

(iv) $v = 15$, $\lambda = 2$.

The base resolution classes are:

$(1_0\ 2_0\ 4_0)$, $(1_1\ 2_1\ 4_1)$, $(1_2\ 2_2\ 4_2)$,
$(3_0\ 3_1\ 0_2)$, $(3_2\ 0_0\ 0_1)$;

and

$(1_0\ 2_0\ 4_0)$, $(1_1\ 2_1\ 4_1)$, $(1_2\ 2_2\ 4_2)$
$(3_1\ 3_2\ 0_0)$, $(3_0\ 0_1\ 0_2)$.

Resolution class generators are:

$(0_2\ 0_0\ 1_1)$, $(0_1\ 1_2\ 1_0)$, $(0_0\ 1_1\ 2_2)$
$(0_2\ 1_0\ 3_1)$, $(0_1\ 2_2\ 3_0)$, $(0_0\ 3_1\ 1_2)$,
$(0_2\ 3_0\ 1_1)$, $(0_1\ 3_2\ 1_0)$, $(0_0\ 4_1\ 3_2)$,
$(0_2\ 4_0\ 3_1)$, $(0_1\ 4_2\ 3_0)$.

(v) $v = 27$, $\lambda = 2$.

The base resolution classes are both the same:

$(1_0\ 4_0\ 6_0)$, $(1_1\ 4_1\ 6_1)$, $(1_2\ 4_2\ 6_2)$,
$(2_0\ 3_0\ 7_0)$, $(2_1\ 3_1\ 7_1)$, $(2_2\ 3_2\ 7_2)$,
$(0_0\ 5_1\ 8_2)$, $(0_1\ 5_2\ 8_0)$, $(0_2\ 5_0\ 8_1)$.

Resolution class generators are omitted.

Case (b): $\lambda = 4$, $v = 6t$.

Again a similar construction can be made. In this case A is the integers modulo 2t. The three pure differences t_0, t_1, and t_2 must only occur twice, while all the other differences, pure and mixed, must occur four times in a base set of paths. The pure differences all occur in four base resolution classes as given in table III, where

$a = [t/2]$, $b = 3a+2$, $c = [(2t-1-b)/3]$.

TABLE III. Base resolution classes for v=6t, $\lambda=4$, $t\geq 3$.

Paths			Differences			
			Pure		Mixed	
1_i	$(2a)_i$	$(2a+2)_i$	2a−1	2	−	
2_i	$(2a-1)_i$	$(2a+3)_i$	2a−3	4	−	
⋮	⋮	⋮	⋮	⋮		
$(a-1)_i$	$(a+2)_i$	$(3a)_i$	3	2a−2	−	
a_i	$(a+1)_i$	$(3a+1)_i$	1	2a	−	
If t is even, the above path should be replaced in two of the classes by:						
a_i	$(a+1)_i$	$(3a+1)_{i+1}$	1		2a	
$(b)_i$	$(b+2c-1)_{i+1}$	$(b+2c+1)_{i+2}$	−		2c−1	2
$(b+1)_i$	$(b+2c-2)_{i+1}$	$(b+2c+2)_{i+2}$	−		2c−3	4
⋮	⋮	⋮			⋮	⋮
$(b+c-1)_i$	$(b+c)_{i+1}$	$(b+3c)_{i+2}$	−		1	2c
0_i	$(2a+1)_{i+1}$	$(b+2c)_{i+2}$	−		2a+1	a+2c+1
If t is odd, the above path should be replaced in two of the resolution classes by:						
0_i	$(2a+1)_i$	$(b+2c)_{i+1}$	2a+1		a+2c+1	
Case 1: b+3c=2t−1. No more paths need be added						
Case 2: b+3c=2t−2. The following four paths should be added, one to each resolution class.						
$(2t-1)_0$	$(2t-1)_1$	$(2t-1)_2$	−		0_{01}	0_{12}
$(2t-1)_2$	$(2t-1)_0$	$(2t-1)_1$	−		0_{20}	0_{01}
$(2t-1)_1$	$(2t-1)_2$	$(2t-1)_0$	−		0_{12}	0_{20}
$(2t-1)_0$	$(2t-1)_1$	$(2t-1)_2$	−		0_{01}	0_{12}
Case 3: b+3c=2t−3. The following two paths should be added to each class, i=1,2,3,4, subscript arithmetic modulo 3.						
Class i:						
$(2t-1)_i$	$(2t-1)_{i+1}$	$(2t-2)_{i+2}$	−		$0_{i\ i+1}$	$(-1)_{i+1\,i+2}$
$(2t-1)_{i+2}$	$(2t-2)_i$	$(2t-2)_{i+1}$	−		$0_{i\ i+1}$	$(-1)_{i+2\,i}$

Once again, all other differences are included in base paths that generate resolution classes as done in theorem 5. Examples:

(vi) $v = 6$, $\lambda = 4$.

Does not fit the form. The base resolution classes, of which there are only 3, are:

$(0_0\ 1_0\ 1_1)$, $(0_1\ 0_2\ 1_2)$;

$(0_1\ 1_1\ 1_2)$, $(0_2\ 0_0\ 1_0)$;

$(0_2\ 1_2\ 1_0)$, $(0_0\ 0_1\ 1_1)$.

The resolution class generators are:

$(0_0\ 0_1\ 0_2)$, $(0_2\ 0_0\ 0_1)$, $(0_1\ 0_2\ 0_0)$,

$(0_0\ 1_1\ 0_2)$, $(0_2\ 1_0\ 0_1)$, $(0_1\ 1_2\ 0_0)$,

$(0_0\ 1_1\ 0_2)$, $(0_2\ 1_0\ 0_1)$, $(0_1\ 1_2\ 0_0)$.

(vii) $v = 12$, $\lambda = 4$.

Does not quite fit exact form. The base resolution classes are:

$(1_0\ 2_0\ 0_0)$, $(1_1\ 2_1\ 0_1)$, $(1_2\ 2_2\ 0_2)$, $(3_0\ 3_1\ 3_2)$;

$(1_0\ 2_0\ 0_0)$, $(1_1\ 2_1\ 0_1)$, $(1_2\ 2_2\ 0_2)$, $(3_2\ 3_0\ 3_1)$;

$(1_0\ 2_0\ 0_1)$, $(1_1\ 2_1\ 0_2)$, $(1_2\ 2_2\ 0_0)$, $(3_1\ 3_2\ 3_0)$;

$(1_0\ 2_0\ 0_1)$, $(1_1\ 2_1\ 0_2)$, $(1_2\ 2_2\ 0_0)$, $(3_0\ 3_1\ 3_2)$.

Resolution class generators are omitted.

(viii) $v = 18$, $\lambda = 4$.

The base resolution classes, repeated twice, are:

$(1_0\ 2_0\ 4_0)$, $(1_1\ 2_1\ 4_1)$, $(1_2\ 2_2\ 4_2)$,

$(0_0\ 3_1\ 5_2)$, $(0_1\ 3_2\ 5_0)$, $(0_2\ 3_0\ 5_1)$;

and

$(1_0\ 2_0\ 4_0)$, $(1_1\ 2_1\ 4_1)$, $(1_2\ 2_2\ 4_2)$,

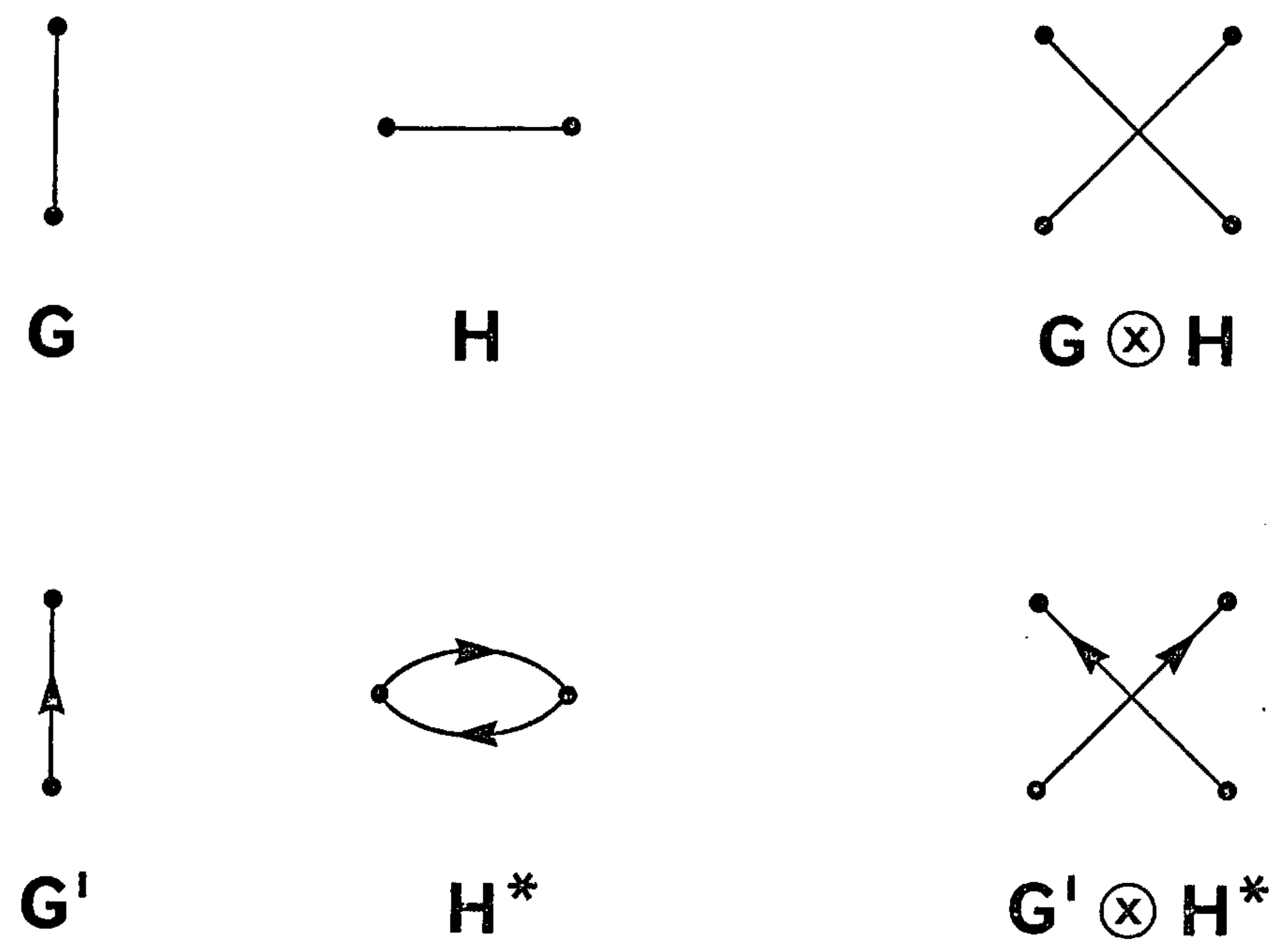
$(0_0\ 3_0\ 5_1)$, $(0_1\ 3_1\ 5_2)$, $(0_2\ 3_2\ 5_0)$.

Resolution class generators are omitted.

6. ACKNOWLEDGEMENTS

I wish to mention the Simon Fraser Combinatorics Seminar in the Spring of 1981 where Pavol Hell mentioned the conjecture concerning categorical products and k-factorizations. This led to my reading reference [4] in detail. I also had some interesting discussions concerning resolvable path designs with Alexander Rosa and Richard Wilson at the University of Waterloo Silver Jubilee Conference on Combinatorics in June, 1982.

List of Figures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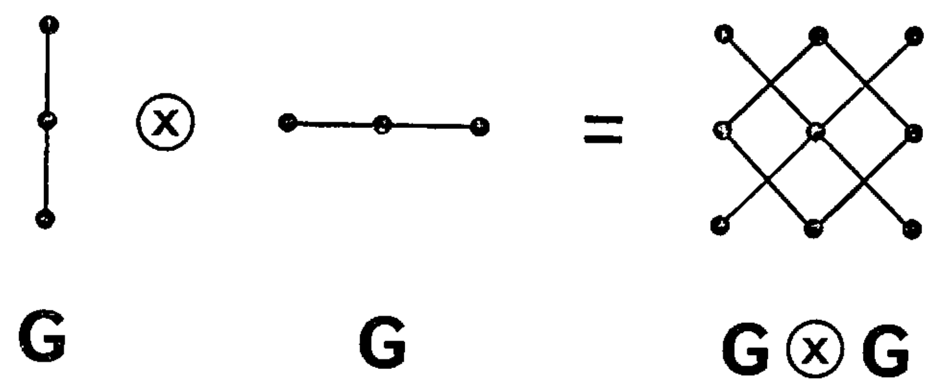
H
G ⊗ H
G'
H*
G' ⊗ H*


G
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G ⊗ G