ORTHOGONAL STARTERS IN FINITE ABELIAN GROUPS

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Abstract

Two problems are considered. First, the conjecture that all odd abelian groups except $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_9$, and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ admit strong starters, is reduced to finding strong starters in five types of groups: the cyclic groups of order 3p, 9p, 3^k for $k > 6$, $5 \cdot 3^k$ for $k > 4$, and $\mathbb{Z}_3 + \mathbb{Z}_{3p}$ where p is any odd prime greater than five. It is shown that all abelian groups G other than $\mathbb{Z}_5$ such that three does not divide the order of G admits a strong starter. As well, strong starters are given in some small non-cyclic groups which were previously not known to admit starters. Also, a multiplication theorem for sets of pairwise orthogonal starters is given. An exhaustive computer search for orthogonal starters in odd groups smaller than 19 is carried out. The results require the construction of special permutations of some groups.

ORTHOGONAL STARTERS IN FINITE ABELIAN GROUPS

J.D. Horton *

1. Introduction

Starters have been very useful in the construction of many types of combinatorial designs. Perhaps the best known design is for round robin tournaments or one-factorizations of the complete graph (see Mendelsohn and Rosa (1985) for a good survey). Strong starters were first used by Stanton and Mullin (1968) to construct Room squares. Since then strong starters have been used to construct Room cubes, Howell designs, Kirkman triple systems, Kirkman squares and cubes, and Kotzig factorizations. (See the papers by Kocay, Stinson and Vanstone (1985), Hwang (1983), Horton (1983)). The major reason strong starters are important is that a strong starter leads to three pairwise orthogonal starters. A set of k pairwise orthogonal starters can be used to construct a Room k-design, or equivalently, k orthogonal one-factorizations of the complete graph, or k pairwise orthogonal symmetric latin squares (see Horton (1981) or Gross, Mullin and Wallis (1973) among others).

In the case of Room squares and related designs, the major direct construction methods use orthogonal starters, and then recursive designs that depend upon the existence of smaller designs are used to solve the existence problems (Mullin and Wallis (1975), Dinitz and Stinson (1981a)). Thus sets of pairwise orthogonal starters play a pivotal role in the construction of many combinatorial designs.

Let G be a finite abelian group of odd order, and let $G^* = G - \{0\}$

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be the set of non-zero elements. A starter for G is a set of unordered pairs of elements of G such that each non-zero element occurs precisely once in some pair, and also precisely once as the difference of one of the pairs. Thus $X = \{\{x_i, y_i\} \mid i = 1, 2, \ldots, (n-1)/2\}$ is a starter if and only if $G^* = \bigcup \{x_i, y_i\}$ and $G^* = \bigcup \{\pm (x_i - y_i)\}$, where the unions are performed for $i = 1$ to $(n-1)/2$.

Although the above is the standard definition, I wish to use a slightly different one which is easily seen to be equivalent. First define D to be a halfset of the group G if:

$$\text{(i)} \quad D \cup -D = G^*, \text{ and}$$

$$\text{(ii)} \quad D \cap -D = \phi.$$

Thus for any nonzero element x in G, either x or -x is in D but not both. Then X is a starter if and only if there is a function $x : D \to G^*$ such that

$$X = \{\{x(d), x(d)+d\} \mid d \in D\}.$$

For X to be a starter, it is necessary and sufficient that the function x satisfies the condition

$$G^* = \bigcup_{d \in D} \{x(d), x(d) + d\}.$$

Given a starter X for a group G of order n, it is well-known how to construct a one-factorization of the complete graph on n+1 vertices, K_{n+1}. Let the vertices of K_{n+1} be the elements of the group G, and an extra fixed point ∞. The set of edges $F_0 = X \cup \{\infty, 0\}$ is a one-factor. Define for each g in G the one-factor $F_g = F_0 + g = \{\{x + g, y + g\} \mid \{x,y\} \in F_0\}$, where the group operation + of G is extended by defining $\infty + g = \infty$. Then the set of all such one-factors $F = \{F_g \mid g \in G\}$ is a one-factorization of K_{n+1}.

Suppose we have two one-factorizations generated by two different starters

$$X = \{\{x(d)\ ,\ x(d) + d\} \mid d \in D\} \text{ and}$$

$$X' = \{\{x'(d)\ ,\ x'(d) + d\} \mid d \in D\},$$

For the two corresponding one-factorizations to be orthogonal it is necessary that $|F_i \cap F'_j| \leq 1$ for any i and j in G. For $i = j$ this statement means, since $\{\infty, i\} \in F_i \cap F'_i$, that $(X + i) \cap (X' + i) = \phi$. Thus $X \cap X' = \phi$, so that $x(d) \neq x'(d)$ for all $d \in D$. For $i \neq j$ the condition $|F_i \cap F'_j| \leq 1$ means that $(X+i) \cap (X' + j) \leq 1$, that is $|X \cap (X' + j - i)| \leq 1$. Hence there must be at most one solution $x(d)$, $x'(d)$ to the equation

$$\{x(d),\ x(d) + d\} = \{x'(d),\ x'(d) + d\} + j - 1.$$

Thus the group elements $x(d) - x'(d)$ must be all distinct. These considerations lead to the definition that X is <u>orthogonal</u> to X' if the elements $x(d) - x'(d)$ are all distinct and non-zero. This definition is equivalent to the definition in Horton (1981).

<u>Problem 1</u>. For a given odd abelian group G, find the maximum number of pairwise orthogonal starters s(G). For a given odd positive integer n define s(n) to be the maximum of s(G) over all abelian groups of order n.

It is easy to show that $s(3) = s(5) = 1$, and that $s(G) \geq 1$ for all odd abelian groups G.

A problem that has been investigated in many papers is to find the maximal number $\nu(n)$ of orthogonal one-factorizations of K_{n+1} (Gross, Mullin and Wallis (1973), Dinitz (1979), Dinitz (1981), Horton (1981)). The above discussion proves:

<u>Theorem 1.1</u>: $\nu(n) \geq s(n)$.

2. Strong Starters

A starter is said to be strong if the sums of all the pairs are distinct and non-zero. If D is a half set and the starter is $X = \{\{x(d), x(d) + d\} \mid d \in D\}$, then the elements $2x(d) + d$ must be distinct and non-zero for X to be strong. The patterned starter P is the starter in which each element is paired with its negative, $P = \{\{g, -g\} \mid g \in G^*\}$. Alternatively if D is a half set, $P = \{\{-d/2, d/2\} \mid d \in D\}$. Then the starter X is orthogonal to the patterned starter P if and only if the elements

$$x(d) - (-d/2) = (2x(d)+d)/2$$

are all distinct and non-zero, that is, X is a strong starter.

Now consider $-X = \{\{-x,-y\} \mid \{x,y\} \in X\} = \{\{-x(d)-d, -x(d)\} \mid d \in D\}$. Now if X is a strong starter, then $-X$ is a strong starter also, since the sums of its pairs are simply the negatives of the sums of the pairs of X, and so $-X$ is orthogonal to P. But $-X$ is also orthogonal to X since $x(d) - (-x(d) - d) = 2x(d) + d$ are all distinct and non-zero. Hence we have proved the well-known result

Theorem 2.1. If G admits a strong starter, then $s(G) \geqq 3$.

This result adds interest to finding the solution to the following variant of problem 1.

Problem 2. What abelian groups admit strong starters?

Quite a bit is known about the existence of strong starters, which were mainly found during the search for Room squares. Before Stanton and Mullin in 1968 exhibited what were effectively strong starters for all odd cyclic groups from 11 to 47, Room squares were thought to be relatively rare. The following results from the literature are the most important for the purposes of this paper.

<u>Theorem 2.2</u>. (Mullin and Nemeth (1969)). Strong starters exist in all finite fields except for those of order of the form $2^k + 1$. The only such prime powers are 9 and the Fermet primes 3,5, 17, 257, 65537, ...

<u>Theorem 2.3</u>: (Chan and Chong, 1974). Strong starters exist all cyclic groups which have order of the form $2^{2^n} + 1$, $n > 2$.

This result left only 3, 5, and 9 as prime powers without strong starters. The four groups of these orders are the only non-abelian groups known not to have strong starters.

The next step was to find strong starters in groups other than the finite fields.

<u>Theorem</u> 2.4. (Horton, 1971). If G admits as strong starter and 3 does not divide the order of G, then $\mathbb{Z}_5 \oplus G$ admits a strong starter.

<u>Theorem 2.5</u>. (Gross and Leonard, 1975). If H is a subgroup of G, H admits a strong starter, G/H admits a strong starter, and there is a permutation π of H such that $\pi + I$ and $\pi - I$ (I is the identity permutation) are also permutations, then K also admits a strong starter.

The condition about the permutation π in the abelian group H seems to be a common one. Let us call a permutation π of a group G <u>strong</u> if both $\pi + I$ and $\pi - I$ are also permutations of G.

<u>Problem 3</u>. What abelian groups admit strong permutations?

The following lemmas include what I know about this problem.

Lemma 2.6. For any prime power q, $q \neq 2$ or 3, the additive group of GF(q) admits a strong permutation.

Proof. Choose an element x in GF(q) such that $x \neq 0$, 1 or -1. Such an x exists if and only if $q \neq 2$ or 3. Define $\pi(g) = xg$ for all g in GF(q). The permutation π is clearly strong.

Lemma 2.7. If G is a group of odd order n, 3 divides n, and the 3-Sylow subgroup of G is cyclic, then G does not admit a strong permutation.

Proof. (Similar to a proof by Wallis and Mullin, 1973). Assume G admits a strong permutation π. Then taking the sums over all group elements i,

$$\begin{aligned} \Sigma\, i^2 &= \Sigma(\pi(i))^2 \\ &= \Sigma(\pi(i)+i)^2 = \Sigma(\pi(i))^2 + \Sigma i^2 + 2\Sigma i\pi(i) \\ &= \Sigma(\pi(i)-i)^2 = \Sigma(\pi(ii)^2 + \Sigma i^2 - 2\Sigma i\pi(i). \end{aligned}$$

Thus, $4\Sigma\, i(\pi(i)) = 0$, so $\Sigma\, i\pi(i) = 0$. Hence, $\Sigma i^2 = 0$.

Now let the 3-Sylow subgroup be cyclic of order k. Then $G = {}_k + H$. Then

$$\sum_{i\varepsilon G} i^2 \equiv |H| \sum_{i\varepsilon\ {}_k} i^2 \qquad \text{(modulo H)}.$$

Since $|H|$ is coprime with k, $\sum_{i\varepsilon\ {}_k} i^2 \equiv 0$ (modulo k)

$$\text{But} \quad \sum_{i\varepsilon\ {}_k} i^2 = \sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$$

$$= (k/3) \cdot (k+1)(2k+1)/2\ .$$

Since k is coprime with k+1, 2k+1, and 2, then $k/3 \equiv 0$ (modulo k), which is impossible.

Lemma 2.8. If G is an abelian group with a subgroup H, and both H and G/H admit a strong permutation, so does G.

<u>Proof</u>. Let $\{a_i | i \varepsilon G/H\}$ be a set of coset representatives of H, let β be a strong permutation of H and let α be a strong permutation of G/H. Define $\pi(\alpha,\beta)$: $G \to G$ by $\pi(a_i + h) = a_{\alpha(i)} + \beta(h)$ for all i in G/H and h in H. That $\pi(\alpha,\beta)$ is a strong permutation is obvious.

Although starters do not involve groups of even order, strong permutations can. For completeness I include the following two lemmas.

<u>Lemma 2.9</u>. An abelian group G that has only one element of order 2, that is its 2-Sylow subgroup is cyclic, does not admit a strong permutation.

<u>Proof</u>. The sum of all the elements of a group is nonzero if and only if there is a unique element of order 2. But summing over all elements i in G, $\Sigma\, i = \Sigma\, \pi\,(i) = \Sigma\,(\pi(i) + i) = \Sigma\, \pi\,(i) + \Sigma\, i$. Thus $\Sigma\, i = 0$. It follows that any abelian group with a cyclic 2-Sylow subgroup does not have a strong permutation.

<u>Lemma 2.10</u>. Any non-cyclic group of order 2^n admits a strong starter.

<u>Proof</u>: Consider the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2k$, $k \geq 2$, and consider the permutation defined by:

$$\pi\,(0,x) = 0,2x \quad \text{for } 0 \leq x < 2^{k-1}\ ,$$

$$\pi\,(0,x) = 1,2x{+}1 \quad \text{for } 2^{k-1} \leq x < 2^k\ ,$$

$$\pi\,(1,x) = 0,2x{+}1 \quad \text{for } 0 \leq x < 2^{k-1}\ ,$$

$$\pi\,(1,x) = 1,2x \quad \text{for } 2^{h-1} \leq x < 2^k\ .$$

It can be easily checked that π is a strong permutation. Any finite non-cyclic group can be factored (in the sense of lemma 2.8) into groups of this form or of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$. Hence lemma 2.10 follows from lemmas 2.6 and 2.8.

One more lemma, which is needed in the proof of the sufficiency of the following list of conditions for the proof of conjecture 1.

Lemma 2.11. The group $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ admits a strong permutation.

Proof. The following strong permutation was found by a computer search:

$\pi(00) = 00$	$\pi(10) = 11$	$\pi(20) = 12$
$\pi(01) = 05$	$\pi(11) = 08$	$\pi(21) = 16$
$\pi(02) = 01$	$\pi(12) = 03$	$\pi(22) = 25$
$\pi(03) = 10$	$\pi(13) = 15$	$\pi(23) = 28$
$\pi(04) = 14$	$\pi(14) = 26$	$\pi(24) = 02$
$\pi(05) = 22$	$\pi(15) = 20$	$\pi(25) = 18$
$\pi(06) = 17$	$\pi(16) = 13$	$\pi(26) = 24$
$\pi(07) = 27$	$\pi(17) = 21$	$\pi(27) = 06$
$\pi(08) = 23$	$\pi(18) = 07$	$\pi(28) = 04$.

Although no more constructions of starters for any other infinite set of groups have been found, Dinitz and Stinson (1981b) published a heuristic hill-climbing algorithm they used to find starters for all odd cyclic groups up to 1000. The algorithm does not always find a strong starter, but in all cases that I know about, it has been restarted and successfully found a strong starter. On the basis of non-rigorous probabilistic arguments, as well as experimentally, the algorithm takes $O(n^2)$ time. Stinson has told me verbally that it has been improved to take $O(n \log n)$ time, under the same non-rigorous conditions. This result leads to the conjectured solution to problem 2.

Conjecture 1: All finite abelian groups of odd order admit a strong starter except $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_9$, and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

To prove the weaker conjecture, that there exist strong starters of all odd orders except 3, 5, and 9, it is necessary and sufficient to prove that there exist strong starters of order 3p and 9p for any prime p greater than 3. To prove conjecture 1 itself, more is required. It is necessary and sufficient to find strong starters in the following groups, where p represents any prime > 5:

(1) $\mathbb{Z}_{3p} = \mathbb{Z}_3 \oplus \mathbb{Z}_p$;

(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_p$;

(3) $\mathbb{Z}_9 + \mathbb{Z}_p$;

(4) all cyclic groups of order 3^a, a = 7,8, ... ;

(5) all cyclic groups of order 5×3^a , a = 4,5, ...

The proof of this sufficiency is straightforward although tedious. As an example of the type of argument required, I shall prove the following easier theorem, which I have not seen published in the literature.

Theorem 2.12. Any abelian group G of odd order n, for which 3 does not divide n, admits a strong starter, as well as a strong permutation.

Proof. The order n is of the form $5^a q$, where q is the product of primes greater than 5. The proof is by induction on a. For a = 0, $n = p_1 p_2 \dots p_k$ where the p_i are primes greater than 5. Then G can be factored into a sequence abelian groups $F_1, F_2, F_3, \dots, F_k$, where $|F_i| = p_i$. By "factored" I mean there exists a sequence of groups $G_0 = G, G_1, G_2, \dots, G_k = \{0\}$ such that $G_{i-1}/F_i = G_i$. By theorems 2.2 and 2.3 each group F_i admits a strong starter. By lemma 2.6 each F_i admits a strong permutation. Applying Theorem 2.5 and Lemma 2.8 repeatedly, each G_i must also admit both a strong starter and a strong permutation. Hence $G_0 = G$ admits a strong starter and strong permutation.

For $a = 1$, $n = 5pq$ where p is a prime greater than 5 and q is a (possibly empty) product of primes greater than 5. Then G has a subgroup H of order 5p. Let $G/H = F$, $H = \mathbb{Z}_5 \oplus \mathbb{Z}_p$, and $|F| = q$. F has a strong starter and a strong permutation by the induction hypothesis, H has a strong starter by Theorem 2.4, H has a strong permutation by lemmas 2.6 and 2.8, and so G has both a strong starter and a strong permutation.

For $a > 1$, $n = 5^2q$ where q is a (possibly empty) product of primes greater than 3 with fewer factors of 5 than n has. Then G has a subgroup H of order 25. Both the abelian groups of order 25 admit strong starters: $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ is the additive group of the finite field GF(25); while Stanton and Mullin (1968) found a strong starter for $\mathbb{Z}_{25}$. Also, any abelian group of order 25 factors into two copies of $\mathbb{Z}_5$, which has a strong permutation. Thus H has a strong permutation. By the induction hypothesis G/H admits both a strong starter and a strong permutation, so G does also.

In order to reduce the list of conditions required to prove conjecture 1 down to the above five, strong starters must also be found in some small groups.

Lemma 2.13. All abelian groups of orders 3^3, 3^4, 3^5, $5 \cdot 3^2$, $5 \cdot 3^3$, $5^2 \cdot 3$, $5^2 \cdot 3^2$ admit strong starters.

Strong starters for all abelian groups of these orders, excluding cyclic groups and Finite Fields, are listed in the appendix. These strong starters were found using the Dinitz and Stinson (1981) hill-climbing algorithm. I did not find the algorithm as reliable as Dinitz and Stinson did. I stopped more than half my runs for taking longer than

I thought reasonable. For some groups, I restarted the algorithm five or six times before it found a starter. In contrast, Dinitz and Stinson had only a ten percent failure rate. A possible reason for the algorithm's poorer performance for me is that the non-cyclic groups contain a lot more small subgroups. Dinitz and Stinson only applied the algorithm in cyclic groups which do not have many small subgroups. The algorithm can get "trapped" in a subgroup by having all or most of the "active elements" coming from a subgroup that has no strong starter. I did not verify this surmise in my computer runs.

3. Larger sets of orthogonal starters

The determination of s(G) and s(n) is a difficult problem. Most of the published results have been lower bounds found by exhibiting sets of orthogonal starters. Although Gross, Mullin, and Wallis (1973) showed that $\nu(2n+1) \to \infty$ as $n \to \infty$, no equivalent result is known for s(2n+1).

The first improvement on the strong starter result was in Horton's thesis (1971b), where s(q) was shown to be at least (q-1)/2 for q a prime power and $q \equiv 3$ (modulo 4). This result was generalized by Dinitz (1979). Although Dinitz's proof itself is not difficult, the following proof is still easier.

Theorem 3.1. (Dinitz 1979) If q is a prime power, $q = 2^k t + 1$, and t is odd, then $s(GF(q)) \geq t$.

Proof. Let w be a generator of $GF(q)^*$ and let $\Delta = 2^{k-1}$. Then $q = 2\Delta t+1$ and $w^{\Delta t} = -1$. Define

$$C_i = \{ w^{2\Delta s+i} \mid s = 0,1,2, \ldots, t-1\}$$

for $i=0,1,\ldots, 2\Delta-1$. Note that $C_i = -C_{\Delta+i}$ and that $C_i = GF(q)^*$. Thus

$$H = C_0 \cup C_1 \cup \ldots \cup C_{\Delta-1}$$

is a half set. Also note that if $a \in C_\Delta$, then $aH=-H$. Define x(d) = d/(a-1), for all d in H. Then H is a starter since: (1) H is a half set; (2) as d runs through H, d/(a-1) runs through the half set H/(a-1); and as d runs through H, d/(a-1) + d = ad/(a-1) runs through the complementary half set aH/(a-1)=-H/(a-1).

Also a and b in C_Δ implies that S_a is orthogonal to S_b, since $d/(a-1) - d/(b-1) = d(b-a)/(a-1)(b-1)$ are all distinct and non-zero for $d \in H$. Thus we have constructed t pairwise orthogonal starters.

Dinitz improved on this bound for numerous small values (1981 and undated). In fact, he suggests that for any prime power $q \equiv 1$ (modulo 4), except $q = 5$, this lower bound can always be improved.

For non-prime powers, other than small numbers, nothing has been published yet. However, the following generalization of Theorem 2.5 is easy to prove. First, we define two permutations α and β of a group G to be <u>orthogonal</u> if $\alpha - \beta$ is also a permutation.

<u>Theorem 3.2</u>. Let G be an odd abelian group with subgroup H. Then

$$s(G) \geq \min \{s(h), s(G/H), p(H)-1\}$$

where p(H) is the maximum number of pairwise orthogonal permutations of H.

<u>Proof</u>: Let $X = \{\{x(d), x(d)+d\} \mid d \in D\}$ where D is a half set, be a starter for H. Let $Y = \{\{H+y(c), H+y(c)+c\} \mid c \in C\}$ where C is a half set of G/H, be a starter for G/H. Let π be a permutation of H such that π-I, where I is the identity, is also a permutation. Define

$$W(X,Y,\pi) = X \cup \{\{\pi(h) - h + y(c), \pi(h) + y(c) + c\} \mid h \in H \text{ and } c \in C\}.$$

$W(X,Y,\pi)$ is a starter since: (1) $D \cup (H + C)$ is a halfset of G; and

$$\begin{aligned}(2)\quad W(X,Y,\pi) &= (\textstyle\bigcup X) \cup \{(\pi-I)h+y(c) \mid h\in H, c\in C\} \cup \{\pi(h)+y(c) \mid h\in H, c\in C\} \\ &= H^* \cup \{H+y(c) \mid c\in C\} \cup \{H+y(c)+c \mid c\in C\} \\ &= G^*.\end{aligned}$$

Now let $X' = \{\{x'(d), x'(d)+d\} \mid d \in D\}$ be a starter for H orthogonal to X, let $Y' = \{\{y(c)+H, y'(c) + c + H\} \mid c \in C\}$ be a starter for G/H orthogonal to Y, and let π' be a permutation of H such that $\pi' - I$ and $\pi' - \pi$ are also permutations of H. Then $W(X,Y,\pi)$ and $W(X',Y',\pi')$ are orthogonal starters for G, since

$(\pi(h) - h + y(c)) - (\pi'(h) - h + y'(c)) = (\pi - \pi')(h) + (y(c) - y'(c))$

are all distinct and not in H for all h in H.

To complete the proof, we must apply the construction to k pairwise orthogonal starters of H, k pairwise orthogonal starters of G/H, and k orthogonal permutations of H such that each permutation is also orthogonal to the identity. But a maximal set of pairwise orthogonal permutations can be modified to include the identity by applying the inverse of one of the permutations to all the permutations.

To apply the theorem, it is necessary to find the appropriate sets of permutations. But these orthogonal permutations can generally be found easily.

Theorem 3.3. For the additive group G of the finite field on q elements, $p(G) \geqq q - 1$.

Proof: For each λ in G, $\lambda \neq 0$, define $\pi_\lambda(h) = \lambda h$. Clearly these q-1 permutations satisfy the required conditions.

Theorem 3.4. If H is a subgroup of abelian group G, then $p(G) \geq \min \{p(H), p(G/H)\}$.

Proof. Let $\{a_i \mid i \in G/H\}$ be a set of coset representatives of H, let β and β' be permutations of H and let α and α' be permutations of G/H. Define $\pi(\alpha, \beta)$ and $\pi(\alpha', \beta')$ as in the proof to lemma 2.8. If $\alpha - \alpha'$ and $\beta - \beta'$ are permutations of G/H and H respectively, then $\pi(\alpha, \beta) - \pi(\alpha', \beta')$ is a permutation of G. Hence sets of k pairwise orthogonal permutations for H and G/H can be used to produce k pairwise orthogonal permutations of G.

We cannot do better than theorem 3.3 for finite fields.

Theorem 3.5. For any finite abelian group, $p(G) \leq |G| - 1$.

Proof: Take a permutation π of a group G and make it the first row of a square $|G|$ by $|G|$ array. For each element of G, add that element to each entry in the first row, and enter the new values into another row. The result will be a latin square. As well, orthogonal permutations will result in orthogonal latin squares. Thus $p(G) \leq |G| - 1$, since the maximum number of pairwise orthogonal latin squares is one less than the length of the side.

4. Computer results

Exact values of s(n) were found for $n \leq 19$ using exhaustive computer search by Joseph Culberson in 1982, under my direction. First an exhaustive search for starters was undertaken in the cyclic group of the appropriate order. Then a graph was defined in which two starters were considered connected if the starters were orthogonal. Finally an exhaustive search for maximal cliques was undertaken in the resulting graph. The major results are summarized in Table 1.

The only non-cyclic group of these orders is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. It is easy to find by hand that there are 9 starters, none of which are orthogonal to any other.

The number of starters seems to be growing more than exponentially, with the ratio between consecutive values being roughly n/3. I have no reason other than this table for believing that this growth will continue for larger values of n.

The value of s(n) was already known to be at least the values in Table I (Gross (1975), Dinitz (1979), and Dinitz (undated)).

I conclude with a table of the known lower bounds s(n) for small n. These are also the known lower bounds for $\nu(n)$ except for n = 9, where $\nu(n \geq 3$. (Dinitz and Stinson, 1981a). I have indicated the earliest source for all values not determined by Theorem 3.1 or the strong starter construction.

Table I. Computer results for small n.

Order of Cyclic Group	Number of Starter	Size of Maximal Clique s(n)	Number of Maximal Clique
3	1	1	1
5	1	1	1
7	3	3	1
9	9	2	6
11	25	5	1
13	133	5	2
15	631	4	144
17	3857	5	72
19	29505	-	-

Table II. Known lower bounds on s(n).

n	s(n)	n	s(n)	n	s(n)
3	1	37	15 **	71	35
5	1	39	5 **	73	9
7	3	41	9 **	75	3
9	2	43	21	77	3
11	5 ***	45	3	79	39
13	5	47	23	81	5
15	4 *	49	3	83	41
17	5 **	51	3	85	3
19	9	53	17	87	3
21	5 **	55	3	89	11 *
23	11	57	3	91	3
25	7 *	59	29	93	3
27	13	61	21 *	95	3
29	13 *	63	3	97	3
31	15	65	3	99	3
33	5 **	67	33	101	31 **
35	5 **	69	3	103	51

* From Dinitz (1979)

** From Dinitz (undated)

*** From Gross (1975)

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APPENDIX

STRONG STARTERS IN SPECIFIC SMALL GROUPS

Z3 + Z9

1 8	0 4	0 5	0 3	1 5	2 7	2 8	1 7	0 7	2 1	1 2	2 2
1 0	0 6	0 1	2 3	0 8	0 2	1 4	1 3	1 1	2 6	2 0	2 4
1 6	2 5										

Z3 + Z27

0 13	0 17	1 14	1 21	0 6	2 12	2 4	1 2
1 1	0 1	1 18	1 5	2 5	2 22	2 2	2 14
2 3	0 11	0 4	1 22	1 6	0 9	2 7	1 17
1 8	2 13	2 25	0 7	0 24	1 0	2 26	2 15
0 19	1 15	0 10	0 5	2 11	1 26	2 10	0 3
2 6	2 0	1 25	1 23	1 19	0 12	0 18	0 21
1 10	2 8	1 13	2 19	2 23	1 24	2 20	1 9
2 16	1 12	2 1	0 23	0 25	2 9	1 4	0 16
0 20	1 3	0 26	0 8	2 24	1 11	1 7	0 15
2 17	2 18	1 20	2 21	1 16	0 2	0 14	0 22

Z9 + Z9

0 7	2 1	3 3	6 3	4 6	5 5	2 5	2 0	1 7	6 5	5 4	8 8
6 6	0 4	7 8	2 8	5 0	1 5	6 0	4 0	2 4	1 8	4 4	7 5
8 5	0 8	1 3	1 1	8 2	1 6	5 2	2 6	4 1	4 7	3 4	4 2
2 7	3 2	0 3	1 0	7 6	3 0	4 5	3 5	3 7	7 1	3 6	5 8
5 1	7 0	0 6	6 7	6 8	0 1	0 5	5 7	8 6	6 1	7 3	2 2
5 3	7 4	5 6	0 2	7 2	4 8	1 4	7 7	8 1	6 4	3 8	8 7
3 1	4 3	2 3	1 2	6 2	8 0	8 4	8 3				

Z3 + Z3 + Z9

2 2 8	1 2 8	2 2 7	1 0 5	2 2 5	2 0 3	0 2 0	2 2 2
1 0 8	2 1 2	2 0 0	2 1 5	0 0 2	0 2 2	1 1 5	0 1 2
0 2 5	1 1 3	1 0 7	0 2 7	0 1 7	1 1 6	2 0 7	0 2 3
2 2 0	0 1 0	1 2 1	0 1 8	0 2 1	0 0 4	0 0 3	2 1 4
1 1 1	1 0 0	1 1 4	1 2 3	2 1 0	2 2 6	2 0 2	1 2 7
2 1 8	0 1 5	2 1 6	1 2 2	2 0 4	0 0 6	1 0 4	0 1 3
2 1 3	0 1 4	1 0 6	2 0 1	1 1 7	0 2 4	2 2 1	0 0 8
1 1 0	1 1 8	1 0 2	2 1 1	0 0 5	0 0 7	1 1 2	2 2 3
0 1 6	1 0 3	0 1 1	1 2 6	2 0 8	2 0 5	0 2 8	1 2 4
1 2 0	1 2 5	2 2 4	2 0 6	1 0 1	2 1 7	0 2 6	0 0 1

Z3 + Z81

0 39	2 0	1 42	2 20	0 26	1 37	0 2	1 67
2 80	1 4	1 2	0 9	1 5	0 29	1 73	1 55
0 48	1 62	1 7	1 36	2 9	2 75	1 45	0 22
0 23	0 66	1 65	2 8	0 42	2 17	2 15	2 59
1 18	2 48	1 49	1 51	0 43	2 44	0 40	0 72
2 72	0 38	2 16	1 8	1 11	0 44	0 17	1 39

2	55	2	34	0	37	1	64	2	74	0	28	0	67	1	59
1	56	1	70	0	27	1	10	2	38	0	80	2	37	2	56
2	73	2	47	2	24	1	76	2	51	1	57	2	54	0	50
0	77	0	57	2	42	0	75	0	65	1	69	1	44	1	72
2	23	2	64	0	5	2	3	1	43	2	18	1	35	1	80
1	54	0	73	0	6	0	70	1	28	1	32	0	54	2	28
2	36	1	20	1	53	2	22	2	10	1	19	0	62	1	79
2	43	0	41	0	1	0	13	2	11	2	70	0	12	1	50
0	18	0	74	0	4	1	75	0	30	2	58	1	24	0	35
2	12	2	5	0	63	0	24	1	34	1	31	1	21	0	36
1	3	0	79	2	40	2	71	0	14	2	35	1	60	2	79
0	51	2	78	0	76	1	27	0	52	1	26	1	71	0	20
1	38	1	25	0	56	2	41	0	58	1	58	2	76	2	46
1	0	2	45	1	47	0	7	1	29	2	65	0	8	2	1
2	26	1	14	1	9	2	27	2	32	1	23	2	29	0	78
2	66	2	33	2	49	2	60	0	71	0	61	0	64	2	30
2	53	1	12	0	32	2	69	1	77	0	25	0	15	1	61
1	74	2	6	2	57	0	19	0	55	2	67	1	17	0	31
1	6	0	3	0	69	2	63	0	47	1	48	1	13	1	78
0	59	1	41	2	77	2	19	2	7	2	31	2	4	2	39
0	49	2	62	2	2	1	46	0	46	2	25	0	33	0	60
1	68	1	63	1	66	2	13	0	21	1	52	1	15	1	16
2	68	0	45	0	11	2	14	1	22	1	30	2	61	2	52
2	50	1	40	0	53	1	33	0	68	0	34	2	21	1	1
0	10	0	16												

Z9 + Z27

5	5	2	19	1	15	8	3	0	20	0	14	8	23	3	6
0	15	2	17	1	6	0	12	6	7	8	6	4	9	0	19
5	4	8	22	6	12	5	6	1	21	2	20	8	8	0	9
4	2	1	9	3	14	8	19	8	5	4	6	5	15	3	11
8	9	7	16	2	10	7	3	4	4	1	2	8	14	6	6
1	1	4	17	4	26	8	7	6	24	5	17	5	14	1	22
7	0	2	6	7	26	0	10	8	13	1	26	0	25	3	25
3	16	4	18	1	24	8	4	3	0	6	21	6	10	3	9
4	13	0	11	3	15	5	25	6	3	7	19	4	25	6	4
7	22	3	17	8	0	2	25	8	10	1	0	7	24	0	4
5	0	4	16	5	22	7	4	6	15	0	22	5	8	3	21
1	7	7	11	1	20	7	17	7	18	7	9	2	21	1	4
0	1	6	16	3	13	3	26	5	10	7	5	3	2	7	25
0	13	3	19	3	5	4	0	6	19	0	23	2	2	1	25
2	4	0	7	3	1	5	24	5	9	5	19	1	16	2	1
7	20	7	12	1	5	8	17	4	12	4	19	4	8	8	11
6	13	5	13	7	13	6	5	6	9	8	1	2	8	7	21
5	1	8	16	8	21	2	16	2	0	3	3	2	24	3	22
6	2	3	12	0	8	5	7	4	23	2	5	2	18	8	26
5	2	6	26	8	18	6	17	1	8	2	22	4	5	4	1
2	15	6	8	7	1	3	7	0	17	5	26	5	12	0	16
8	20	8	15	4	20	1	11	3	4	3	20	2	7	2	9
1	19	6	22	5	20	2	23	4	7	7	15	8	2	7	14
4	14	8	25	6	25	7	8	5	21	0	3	0	18	1	14
8	12	3	24	6	23	6	20	6	1	4	3	4	24	2	3
3	18	5	23	0	6	8	24	5	18	1	3	7	2	4	15

7	10	2	26	3	8	4	21	7	6	5	3	2	11	6	11
4	10	4	22	6	18	0	2	3	23	1	23	6	14	0	24
7	7	1	12	1	18	5	16	2	14	2	13	1	13	0	21
7	23	3	10	1	10	0	5	2	12	5	11	0	26	1	17
6	0	4	11												

Z3 + Z3 + Z27

1	0	22	0	1	21	0	2	7	0	1	2
0	0	23	0	1	8	1	2	1	2	0	7
2	2	25	0	2	14	2	1	19	1	1	24
1	1	10	2	2	1	2	2	12	2	0	23
0	0	3	2	2	3	0	2	23	0	1	23
2	1	7	1	2	21	2	1	4	2	1	2
0	0	9	2	2	17	2	2	9	2	2	19
0	0	5	1	2	26	2	1	12	0	2	1
1	2	15	2	0	13	0	2	0	2	1	15
2	2	0	1	2	22	2	1	5	2	1	20
1	0	10	0	1	12	1	1	19	0	0	8
1	2	24	1	1	5	0	1	20	2	2	11
1	2	23	1	1	7	2	2	23	2	2	2
1	1	3	1	2	9	0	1	24	0	1	6
0	2	9	2	0	21	1	1	26	2	1	14
1	0	24	0	0	1	0	2	15	1	1	21
0	1	19	0	2	5	2	2	24	1	0	2
0	1	5	2	2	5	2	2	6	2	0	8
0	1	15	2	0	12	2	0	4	2	1	0
0	0	2	2	0	25	2	1	17	0	2	16
1	1	1	1	0	20	2	0	5	1	1	9
1	1	11	0	2	22	1	2	17	0	0	7
2	1	26	1	1	25	0	0	4	1	0	3
2	0	24	2	2	26	0	2	25	2	0	11
1	2	5	2	0	26	0	1	13	0	1	26
0	0	21	0	2	20	0	1	0	0	0	20
1	1	18	1	1	2	1	2	14	2	0	19
0	0	6	2	0	3	0	0	18	2	1	10
2	2	20	1	1	0	1	2	11	1	0	4
1	0	16	0	0	10	2	2	13	1	0	14
2	1	25	1	0	12	1	2	4	0	2	17
0	1	14	0	2	4	0	2	8	0	1	25
1	0	5	0	1	1	0	2	2	1	1	14
2	0	14	2	1	10	2	0	16	1	0	26
1	1	15	1	0	18	2	2	7	2	0	10
2	1	3	1	0	6	2	2	18	2	1	9
0	2	18	0	2	11	2	0	15	0	1	16
2	1	11	1	2	18	1	2	7	1	1	13
0	2	10	0	1	22	2	0	18	0	0	26
2	1	24	1	2	6	0	0	13	1	0	25
1	1	4	1	0	9	0	1	11	1	0	8
1	0	13	1	1	12	2	0	20	1	2	16
2	2	16	2	2	8	1	2	8	0	2	26
1	0	0	1	0	23	2	0	1	0	2	3
1	0	15	0	2	13	1	0	7	2	1	21
1	1	16	0	2	24	0	2	19	2	0	2

2	1	22	0	2	12	2	2	15	0	0	24
1	1	17	2	0	22	1	2	13	0	1	18
1	2	25	0	1	17	0	0	25	2	2	10
0	0	11	1	0	11	0	2	6	2	2	4
0	1	4	1	1	22	2	1	6	2	1	1
0	0	12	1	1	8	1	0	19	0	0	22
2	0	17	1	1	6	2	1	8	0	0	15
0	0	14	1	0	21	1	1	23	1	1	20
1	0	17	2	0	9	2	1	16	0	1	9
0	0	16	0	1	3	0	1	7	2	0	0
1	2	3	1	2	2	2	1	13	1	2	10
2	0	6	0	0	19	0	1	10	1	2	20
1	2	0	2	2	21	2	2	14	2	1	23
1	2	19	0	2	21	2	2	22	1	2	12
0	0	17	1	0	1						

Z3 + Z9 + Z9

2 6 2	2 1 3	2 7 8	2 1 0	1 6 1	1 4 4	0 3 4	0 8 8
1 5 3	1 7 5	2 6 7	2 1 5	2 2 6	2 6 6	2 8 8	0 2 4
0 7 8	1 2 5	2 6 1	2 8 5	0 7 0	2 0 8	2 4 4	0 1 1
0 6 3	0 4 0	2 3 2	2 3 7	1 7 8	0 6 4	2 0 3	2 0 4
1 0 6	2 2 4	2 1 6	0 5 2	1 6 3	2 2 1	0 0 4	2 5 5
1 0 8	1 5 2	0 1 3	1 2 1	2 0 5	2 1 2	2 3 3	0 6 6
1 0 1	0 4 6	2 0 1	0 7 3	0 7 4	2 6 3	2 8 1	0 5 0
1 6 6	0 3 8	0 3 3	2 1 8	2 1 4	1 2 2	0 5 6	0 6 1
1 4 8	0 2 0	1 8 2	2 7 1	1 2 7	0 8 3	1 5 8	1 2 6
0 8 4	1 6 0	1 8 4	0 3 5	2 6 4	2 5 1	0 6 5	0 1 4
2 6 5	0 2 8	0 5 3	2 8 0	2 0 6	1 3 1	2 8 3	1 1 4
2 5 2	2 2 8	0 4 8	0 6 7	1 8 0	1 3 3	0 5 8	0 5 1
2 4 8	1 5 7	0 0 6	1 3 5	0 1 8	0 4 5	1 4 2	0 4 7
2 4 0	2 5 0	2 8 7	2 2 5	2 2 3	0 7 5	0 8 2	2 7 0
2 6 8	2 1 1	1 0 2	0 5 7	0 2 1	2 4 3	0 0 8	1 3 0
0 7 7	2 3 4	0 3 6	2 3 8	1 6 4	2 5 4	2 4 7	2 7 3
1 4 6	1 6 2	1 2 0	0 2 6	1 8 1	2 5 3	0 3 7	0 1 0
1 7 1	0 5 5	2 7 2	0 2 2	2 4 2	0 8 0	2 2 0	0 6 2
0 3 0	1 8 6	1 1 3	2 2 2	1 6 5	2 8 6	1 1 0	0 2 3
2 6 0	1 0 0	0 3 2	1 0 3	1 2 4	2 5 6	0 0 5	1 1 8
1 8 5	0 0 2	0 0 7	1 8 3	1 5 5	0 5 4	2 3 5	1 1 5
1 7 2	0 4 2	0 0 3	1 5 4	2 0 2	2 8 4	2 1 7	0 8 7
1 7 7	2 7 7	1 5 1	0 0 1	0 4 3	1 1 7	2 7 6	2 4 6
2 4 5	1 6 8	2 3 6	2 3 0	0 3 1	2 5 7	0 8 1	1 8 7
2 7 4	1 4 7	0 6 8	2 2 7	1 7 4	1 2 8	1 3 4	1 4 3
1 4 0	0 2 7	1 0 4	2 8 2	1 6 7	0 4 4	0 4 1	2 3 1
2 4 1	0 2 5	0 1 6	1 1 2	1 1 1	1 7 6	0 1 5	0 8 5
1 4 1	1 3 8	2 7 5	0 6 0	1 8 8	0 7 2	1 2 3	0 1 7
0 8 6	0 7 1	1 5 6	1 4 5	1 3 7	1 1 6	2 0 0	1 7 3
2 5 8	1 5 0	1 3 2	0 7 6	1 7 0	0 1 2	1 3 6	1 0 7
2 0 7	1 0 5						

Z3 + Z3 + Z3 + Z9

2 1 0 2	1 1 1 8	2 1 0 3	0 0 2 8
0 0 2 3	0 0 0 7	2 2 1 7	2 1 0 1

1	1	2	0	2	0	2	2	2	0	1	2	2	2	0	1
2	1	1	6	2	1	0	0	2	2	1	1	0	2	0	0
2	0	0	4	1	0	2	8	1	2	2	7	2	0	0	3
1	1	1	4	1	2	0	7	1	2	1	2	2	1	1	2
1	1	0	0	2	0	1	0	2	2	2	3	2	0	2	5
2	0	1	4	2	2	1	0	0	2	0	7	1	1	2	1
1	2	2	8	0	2	1	1	0	1	1	1	0	2	2	3
0	2	0	4	0	0	1	4	1	2	1	4	1	2	1	0
0	1	2	2	1	1	2	8	0	2	1	5	0	2	2	6
0	1	2	1	0	0	0	3	1	1	0	5	0	0	2	6
2	0	1	3	1	1	0	2	2	1	2	8	2	2	0	7
2	1	2	0	0	0	2	4	0	1	0	1	2	0	2	7
1	0	0	3	2	1	0	5	0	0	0	2	2	0	1	7
1	1	0	8	0	2	1	4	2	2	2	7	0	2	2	2
0	1	2	7	2	2	2	8	1	2	1	8	1	0	1	2
1	2	1	7	0	2	1	2	0	0	2	7	0	0	1	8
2	2	1	2	1	0	0	5	2	0	2	3	1	2	0	3
2	0	0	2	1	0	2	3	2	1	0	6	2	2	2	1
2	2	1	3	1	0	2	2	1	0	1	7	0	0	2	2
1	2	2	5	2	2	0	5	1	0	0	6	0	1	2	8
1	2	1	3	0	1	1	6	2	2	2	5	1	0	1	3
2	1	2	1	0	1	1	8	1	1	0	6	0	2	1	7
0	0	0	5	0	0	1	7	2	0	0	0	0	2	1	8
1	0	1	5	1	0	0	7	2	1	2	5	2	2	0	8
1	0	0	0	0	1	1	0	0	0	1	5	2	0	0	8
2	0	0	6	1	0	2	5	2	2	1	6	1	2	2	4
2	2	1	8	0	0	2	0	0	2	0	5	2	2	2	2
0	2	2	7	2	2	2	4	1	1	1	2	0	0	1	6
0	2	2	4	0	0	1	3	2	1	2	3	1	0	2	0
0	0	2	5	1	0	2	6	1	2	0	2	1	1	2	7
1	2	1	1	2	0	2	1	2	2	0	4	1	1	1	5
1	1	2	5	2	2	0	2	0	1	0	2	1	2	2	0
0	1	0	0	2	1	2	7	2	1	1	7	1	1	2	6
0	0	2	1	1	1	0	3	2	0	0	5	0	2	1	0
2	2	1	5	1	2	1	6	1	1	1	7	2	2	0	0
2	1	1	1	1	1	1	1	2	1	0	4	0	2	2	8
1	1	2	3	2	2	1	4	0	1	1	4	0	1	1	3
0	1	0	4	0	1	2	4	1	2	0	6	0	1	0	5
2	1	0	7	0	2	0	2	0	0	0	1	0	1	2	6
1	1	0	1	2	1	0	8	2	1	1	3	0	1	1	5
1	0	1	1	0	2	2	5	2	2	2	6	2	0	1	6
0	0	1	0	0	2	1	3	0	2	0	8	1	0	1	6
2	0	0	1	2	0	0	7	1	1	1	3	1	1	0	7
2	1	1	8	0	2	1	6	0	0	0	8	1	2	2	1
0	2	0	6	1	1	0	4	1	2	0	0	1	2	2	6
2	0	2	0	2	2	0	3	1	2	2	2	2	1	1	0
1	0	0	2	1	2	0	4	1	1	1	6	0	2	0	3
0	0	1	1	1	1	1	0	1	0	1	8	0	1	1	7
2	0	2	8	1	1	2	2	1	2	2	3	1	0	0	8
1	0	0	1	2	0	1	5	0	0	0	6	0	1	0	6
0	1	2	3	1	0	1	0	2	1	1	4	2	2	0	6
0	0	0	4	0	1	0	3	1	1	2	4	0	1	0	7
0	1	2	5	1	0	2	1	0	2	0	1	1	0	2	7
1	2	0	5	0	1	1	2	1	0	1	4	0	1	0	8

1 2 0 8	1 2 0 1	2 0 1 8	2 1 2 6
1 2 1 5	0 0 1 2	2 0 2 6	2 1 2 2
0 1 2 0	0 2 2 1	1 0 2 4	2 1 2 4
1 0 0 4	2 0 2 4	2 0 1 1	2 2 2 0
0 2 2 0	2 1 1 5		

Z3 + Z15

2 6	0 13	1 13	2 10	0 2	1 6	0 6	0 8
2 14	1 3	0 3	0 4	2 9	2 2	0 9	2 7
1 5	0 5	0 10	2 1	0 14	2 8	0 7	0 12
1 2	0 1	2 0	1 10	2 4	1 1	1 4	2 12
2 5	2 11	0 11	1 9	1 7	1 11	1 12	1 0
1 8	2 3	1 14	2 13				

Z3 + Z45

2 7	0 25	2 3	1 38	0 5	0 11	1 44	2 38
0 28	0 37	0 12	1 18	0 23	2 14	2 0	2 18
1 37	1 2	0 22	0 14	0 40	0 38	2 12	0 29
0 34	1 8	2 32	0 8	2 41	0 3	2 29	2 43
0 4	2 27	2 21	2 26	2 28	1 12	2 1	0 2
1 4	2 6	0 36	0 7	1 29	0 9	0 18	2 37
2 31	0 20	1 40	2 36	2 33	2 44	0 21	0 42
1 19	0 41	1 28	1 6	1 35	1 34	2 11	0 1
2 24	2 4	2 19	1 7	2 10	1 15	0 31	1 0
2 17	0 30	0 24	1 11	0 44	1 42	1 24	1 21
2 42	1 14	1 25	0 33	1 22	2 13	1 39	1 13
1 36	2 22	2 15	2 30	1 43	0 16	0 17	1 17
2 8	2 25	2 20	1 32	1 20	1 33	1 16	0 32
1 10	1 3	1 5	0 6	1 30	2 9	1 31	0 26
1 9	2 39	1 1	2 5	0 27	0 15	2 35	1 27
0 13	2 2	1 23	0 43	2 40	0 10	0 39	0 35
2 16	0 19	2 23	1 26	1 41	2 34		

Z3 + Z3 + Z15

2 2 12	0 2 6	0 0 2	0 2 7
0 1 1	2 2 14	1 2 2	2 2 8
0 2 0	1 0 10	2 0 12	1 2 8
0 1 9	1 1 10	1 1 0	1 1 7
1 2 4	1 1 13	1 1 14	2 0 9
1 2 11	0 1 8	1 0 14	1 0 0
2 0 1	2 0 5	2 1 10	1 0 5
1 0 1	1 2 12	1 2 5	2 2 1
2 1 1	1 2 1	2 0 14	0 1 13
2 2 5	2 2 3	2 0 4	0 0 12
0 0 10	0 1 2	1 1 5	1 0 8
0 0 3	2 2 7	1 1 6	2 1 13
1 0 11	0 0 1	0 0 13	1 2 10
2 2 13	2 1 0	2 1 4	2 0 10
0 1 6	2 1 3	1 1 4	2 0 0
1 1 12	1 1 3	0 0 4	1 2 3
0 1 10	2 1 8	2 1 5	0 0 6

2	0	8	0	2	12	1	1	11	2	2	2
1	0	13	2	0	13	1	1	9	0	1	11
0	1	3	2	0	3	0	2	13	0	1	12
0	1	7	0	0	11	0	0	7	1	2	14
0	0	5	1	2	13	2	2	10	0	2	14
1	0	6	2	2	0	0	2	8	0	0	8
1	0	4	0	0	14	1	0	12	0	2	4
2	0	2	2	1	7	0	1	0	2	2	9
0	1	4	1	0	2	1	0	7	2	1	9
0	1	14	2	2	11	0	2	1	2	0	11
0	2	3	1	2	0	1	1	8	2	2	6
0	2	10	1	2	9	0	1	5	1	2	6
0	0	9	1	1	1	1	0	3	1	1	2
2	0	6	2	1	14	1	2	7	1	0	9
2	1	2	0	2	11	2	1	11	2	1	6
0	2	2	0	2	5	2	1	12	0	2	9
2	0	7	2	2	4						

Z5 + Z15

1	9	4	12	3	2	4	2	1	14	2	5	3	11	1	1
0	6	2	11	4	4	4	1	4	10	2	10	4	11	0	5
1	7	1	8	1	12	0	9	2	1	4	7	4	14	3	13
2	6	3	8	1	3	0	8	4	9	1	2	4	5	4	3
3	6	0	4	3	10	0	12	3	3	2	13	3	7	1	6
1	5	0	13	2	4	0	10	2	7	2	14	1	11	1	0
2	9	4	8	0	2	4	6	0	7	0	1	3	5	4	13
4	0	0	14	3	12	2	0	1	13	3	9	0	3	3	0
0	11	3	4	2	12	3	1	2	3	2	8	1	10	3	14
2	2	1	4												

Z3 + Z75

1	11	0	41	2	45	2	65	0	22	2	10	2	5	1	48
1	69	1	35	1	67	1	58	0	8	2	12	2	1	0	12
0	18	1	26	1	65	2	54	2	71	1	32	0	67	0	28
2	9	2	69	0	35	2	38	2	61	0	13	0	54	2	72
2	18	2	51	2	7	1	51	2	21	1	19	1	74	0	31
0	40	0	24	1	52	0	47	2	57	2	43	1	68	1	12
0	26	0	52	2	8	2	58	1	54	1	61	2	24	2	30
1	47	0	38	1	59	1	30	1	55	1	72	0	11	1	62
1	4	2	32	2	70	0	20	0	25	2	67	0	48	1	40
0	2	1	9	0	72	2	31	0	63	0	64	0	51	0	16
2	68	1	39	1	24	0	69	1	0	2	14	2	33	0	50
1	63	0	19	0	15	2	34	1	16	0	1	1	44	2	60
0	53	1	2	1	18	2	19	2	36	2	48	1	56	2	3
1	6	2	4	2	29	1	45	1	53	1	1	1	36	2	74
1	46	0	68	1	38	0	65	2	26	0	44	0	27	1	10
1	70	1	66	1	23	2	2	2	52	2	49	1	34	1	7
0	58	2	55	2	35	1	31	0	14	0	9	1	60	0	37
1	41	0	66	0	56	0	45	0	34	0	21	2	50	0	49
2	41	0	36	1	71	2	15	0	29	2	64	2	73	0	59
2	56	0	23	1	13	1	43	1	14	2	63	1	20	0	61
2	25	1	37	1	22	2	16	2	20	1	27	2	66	2	42

1	5	2	11	0	55	1	15	0	6	0	43	1	57	0	5
0	30	2	17	1	25	0	4	0	46	2	59	0	62	0	70
2	53	0	33	1	64	1	33	2	47	0	73	2	44	2	46
0	32	0	60	0	74	2	37	0	71	1	42	2	39	1	3
2	23	2	13	1	8	1	29	0	39	0	57	0	42	2	22
1	49	1	17	2	27	0	17	0	7	1	73	2	62	2	40
0	3	1	50	2	28	1	28	2	0	0	10	1	21	2	6

Z5 + Z45

2	23	1	36	1	12	0	35	3	38	1	18	4	2	0	33
0	9	2	15	0	17	2	28	4	20	3	28	1	41	0	18
0	32	0	11	4	15	2	20	1	37	4	23	3	15	2	1
1	27	1	13	0	3	1	21	0	44	1	10	2	19	0	36
1	8	1	1	3	14	1	11	3	42	0	26	1	26	1	38
0	21	3	12	3	32	2	36	0	6	2	22	2	10	2	6
4	25	3	6	3	1	0	24	0	42	3	16	3	30	0	12
1	30	2	43	0	28	4	27	1	5	4	6	2	33	2	7
4	38	4	44	1	16	0	40	3	3	4	12	3	34	2	37
1	24	3	43	1	29	1	0	3	27	1	42	3	25	2	41
1	6	0	39	1	22	2	26	1	4	2	39	3	22	1	20
4	36	0	43	3	40	1	28	3	11	3	20	4	37	0	37
0	2	0	27	1	19	4	9	0	38	4	8	2	24	2	21
0	14	3	10	2	27	2	12	4	21	3	13	1	15	4	42
2	14	1	43	2	2	3	4	0	31	1	3	4	40	4	5
3	26	4	41	3	0	0	41	0	29	2	8	2	18	2	35
0	30	1	9	2	34	4	14	1	23	4	1	4	18	2	25
4	26	1	33	3	37	2	4	3	33	2	13	2	38	0	23
0	7	3	39	2	3	4	3	0	34	4	39	0	20	0	22
4	22	0	4	1	31	3	18	3	21	3	8	4	43	4	32
1	14	4	24	3	5	4	33	0	16	3	44	1	2	2	0
3	17	3	9	3	23	2	30	4	4	4	31	4	29	4	7
4	30	1	35	2	32	0	1	3	2	2	42	1	7	0	13
2	29	3	35	0	10	2	31	3	36	4	17	1	34	4	0
1	32	4	35	1	40	3	41	3	31	4	11	1	44	2	9
4	13	2	5	0	15	4	16	2	44	0	8	0	19	2	11
1	25	3	19	2	17	2	16	0	25	4	34	0	5	1	39
3	29	3	24	4	28	2	40	3	7	4	10	4	19	1	17

Z15 + Z15

11	9	0	8	3	4	0	10	10	5	4	10	3	11	9	0
0	1	10	13	9	14	1	11	7	13	10	2	0	2	1	14
8	8	2	8	7	14	4	11	3	6	2	7	9	11	10	3
14	13	0	3	9	7	3	14	5	9	4	6	2	12	8	9
2	10	13	6	9	12	8	6	9	2	9	9	9	1	7	9
9	3	14	7	12	8	6	7	10	7	14	9	8	12	1	9
14	14	11	1	6	6	11	4	1	2	6	4	2	14	7	0
6	3	14	12	11	6	4	5	11	3	6	8	1	13	5	3
4	0	4	2	12	7	10	1	11	0	3	7	5	0	10	14
6	11	3	9	14	4	3	0	11	14	7	11	13	9	1	0
8	11	6	0	11	5	8	5	9	6	2	13	8	1	7	3
3	10	2	1	1	10	11	13	5	8	14	1	6	1	2	3
12	10	14	10	14	2	6	12	0	4	1	8	4	14	12	3

7	2	7	8	7	12	10	11	0	13	5	5	13	14	9	4
5	1	11	7	3	12	3	13	13	8	1	4	4	3	2	4
5	12	8	2	7	5	10	0	1	5	7	4	4	1	3	8
9	13	13	13	5	6	5	11	7	10	9	5	13	1	6	2
0	11	6	9	5	14	10	8	10	9	5	4	2	9	0	12
3	1	12	5	13	12	11	10	12	9	3	3	8	3	6	10
5	10	10	10	0	6	7	6	14	0	5	2	4	12	4	8
13	7	10	6	12	14	14	8	11	8	2	11	12	12	11	11
11	12	10	12	8	10	11	2	2	6	3	2	12	13	4	7
13	10	6	5	14	5	2	2	7	7	12	0	6	13	1	7
12	4	1	12	12	6	8	14	8	4	4	13	7	1	6	14
9	8	13	5	0	14	2	0	14	11	3	5	14	6	9	10
0	9	13	4	4	9	1	1	4	4	12	2	8	7	1	3
0	5	13	2	8	13	14	3	2	5	0	7	12	1	13	11
1	6	5	7	10	4	8	0	12	11	5	13	13	3	13	0

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