APPLICATIONS OF THE EXTENSION PRINCIPLE TO THE PLANE

by

Xiubin Xu and Bradford G. Nickerson

TR95-092, March 1995

Faculty of Computer Science
University of New Brunswick
Fredericton, N.B. E3B 5A3
Canada

Phone: (506) 453-4566
Fax: (506) 453-3566

Applications of the Extension Principle to the Plane

University of New Brunswick

Faculty of Computer Science

Technical Report TR95-092

Xiubin Xu

xiubin@halmos.math.unb.ca

Department of Mathematics and Statistics

and

Bradford G. Nickerson

bgn@unb.ca

Faculty of Computer Science

University of New Brunswick

P.O. Box 4400, Fredericton, New Brunswick

CANADA E3B 5A3

March, 1995

1 Introduction

The extension principle provides a general method for extending nonfuzzy mathematical concepts in order to deal with fuzzy quantities. It was introduced by Zadeh in 1975 [6]. With this principle, it is possible to fuzzify any domain of mathematical reasoning based on set theory. As in Gaines [3] or [1, P.38], "the fundamental change is to replace the precise concept that a variable has a value with the fuzzy concept that a variable has a degree of membership to each possible value."

Let's introduce the idea of this principle. Let X be a Cartesian product of universes, $X = X_1 \times X_2 \times \cdots \times X_r$, and $A_1, A_2, \ldots, A_r$ be r fuzzy sets in $X_1, X_2, \ldots, X_r$, respectively. The Cartesian product of $A_1, A_2, \ldots, A_r$ is defined as

$$A_1 \times \cdots \times A_r = \int_{X_1 \times \cdots \times X_r} \min(\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r))/(x_1, \ldots, x_r).$$

Let f be a mapping from X to a universe Y such that $y = f(x_1, \ldots, x_r)$. The extension priciple allows us to induce from r fuzzy sets A_i a fuzzy set B on Y through f such that

$$\mu_B(y) = \sup_{y=f(x_1,\ldots,x_r)} \min(\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)). \tag{1}$$

$$\mu_B(y) = 0, \qquad \text{if } f^{-1}(y) = \emptyset,$$

where $f^{-1}(y)$ is the inverse image of y. $\mu_B(y)$ is the greatest among the membership values $\mu_{A_1\times\cdots\times A_r}(x_1,\ldots,x_r)$ of y using $r-$tuples $(x_1,\ldots,x_r)$. Zadeh usually writes (1) as

$$B = f(A_1,\ldots,A_r) = \int_{X_1\times\cdots\times X_r} \min(\mu_{A_1}(x_1),\ldots,\mu_{A_r}(x_r))/f(x_1,\ldots,x_r),$$

where the sup operation is indicated by union (i.e. by the $\int$ sign).

The compatibility of the extension principle with $\alpha - cuts$ can be stated as

Theorem 1

$$[f(A_1,\ldots,A_r)]_\alpha = f(A_{1\alpha},\ldots,A_{r\alpha})$$

if and only if $\forall y \in Y, \exists x_1^*,\ldots,x_r^*$, *such that*

$$\mu_B(y) = \mu_{A_1\times\cdots\times A_r}(x_1^*,\ldots,x_r^*),$$

i.e., the upper bound in (1) is attained.

By changing sup or min in (1) into other operations, we can get other extension principles. For example, Jain [4] proposed replacing sup in (1) by the probabilistic sum $\hat{+}$ ($u\hat{+}v = u + v - uv$). The nationale behind this operator is that the membership of y in $f(A_1, \ldots, A_r)$ should depend on the number of r-tuples $(x_1, \ldots, x_r)$ such that $y = f(x_1, \ldots, x_r)$. This extension principle sounds more probabilistic than fuzzy, particularly if we also replace min by product. It has been pointed out by Dubois and Prade [2] that, in general, $f(A_1, \ldots, A_r)$ is a classical subset of y when $X = \mathbb{R}$ (with min or product) and continuous membership functions are considered. So the result depends only on the supports of the A_i, which invalidates this principle as one of fuzzy extension. Another extension principle is discussed in [1], which can be obtained by just replacing min by product in (1). This principle implicitly assumes some "interactivity" or possible "compensation" between the A_i. The problem of interactivity is considered in [1, pp. 277 - 296]. It does not seem that this latter principle has the same drawbacks as does that of Jain.

In this report, we will discuss the applications of the extension principle to the normal plane $\mathbb{C}$. We will give four definitions of fuzzy points in the second part. Let $\tilde{\mathbb{C}}$ be the set of all the fuzzy points. We will discuss some geometric properties of $\tilde{\mathbb{C}}$ in the third part and some analytic and topological properties of this set in the fourth part. Before going into our work, we'd like to introduce the concept of fuzzy distance $\tilde{d}$ of two fuzzy sets A and B (see

[1]): Let X be a metric space equipped with the pseudometric d, and A, B be two fuzzy sets on X, then the fuzzy distance between A and B is defined using (1) as

$$\forall \delta \geq 0, \qquad \mu_{\tilde{\delta}(A,B)}(\delta) = \sup_{d(u,v)=\delta} \min(\mu_A(u), \mu_B(v)).$$

2 Definitions of Fuzzy Points

The first definition of fuzzy point is a generalization of the single point in the plane.

Definition 1 *For a fuzzy set $A \subset \mathbb{C}$, if there exists $u \in A$, such that $\mu_A(u) = \lambda$ for some $\lambda \in [0,1]$, and $\mu_A(v) = 0$ for $v \neq u$, then we call the fuzzy set A a fuzzy point, we can express it as A_λ. (Do not make confusion with $\alpha-$cuts of a fuzzy set, we do not discuss $\alpha-$cuts here.)*

When $\lambda = 1$, the fuzzy point is really a crisp point.

We can introduce fuzzy point from the concept of fuzzy number.

Definition 2 *If A is the Cartesian product of two fuzzy numbers, then we call A a fuzzy point on the plane.*

More generally, we can define a fuzzy point as

Definition 3 *A fuzzy point is a convex normalized fuzzy set A of the plane* $\mathbb{C}$ *such that*

(a) $\exists$ *exactly one* $c_0 \in \mathbb{C}$ *satisfying* $\mu_A(c_0) = 1$, c_0 *is called the mean value of A),*

(b) μ_A *is piecewise continuous.*

The following theorem states the relations between these three definitions:

Theorem 2 *(a) Fuzzy points defined by Definition 1 with* $\lambda = 1$ *are fuzzy points in the meaning of Definition 2 and Definition 3.*

(b) Fuzzy points given by Definition 2 satisfy the conditions in Definition 3.

Proof. (a) is obvious, we only prove (b) here.

Suppose A is the Cartesian product of two fuzzy numbers on $\mathbb{R}$: $A = X \times Y$, then by the definition of fuzzy number and Cartesian product of fuzzy sets, $\exists! u = (x, y) \in \mathbb{C}$, such that $\mu_X(x) = 1$, $\mu_Y(y) = 1$, hence $\mu_A(u) = 1$, i.e., A is normalized and satisfies (a) in Definition 3, where $\exists!$ means that there exists one and only one. (b) of Definition 3 is satisfied obviously, we only need to prove A is convex.

$\forall u = (a, b), v = (c, d) \in \mathbb{C}$ and $\alpha \in [0, 1]$,

$$\mu_A(\alpha u + (1-\alpha)v) = \min(\mu_X(\alpha a + (1-\alpha)c), \mu_Y(\alpha b + (1-\alpha)d))$$

$$\geq \min(\min(\mu_X(a), \mu_X(b)), \min(\mu_Y(c), \mu_Y(d)))$$

$$= \min(\min(\mu_X(a), \mu_Y(c)), \min(\mu_X(b), \mu_Y(d)))$$

$$= \min(\mu_A(u), \mu_A(v)).$$

Hence A is a fuzzy point in the meaning of Definition 3.

The following results are about the fuzzy distance of the above fuzzy points.

Lemma 1 *If $S \subset [0,1]$, then $\tilde{+}_{s\in S} \in [\sup_{s\in S}, 1]$. Especially, if $1 \in S$, then $\tilde{+}_{s\in S} = 1$.*

Proof. $\forall x, y \in S$, $x\tilde{+}y = x + y - xy = x + (1-x)y \in [y, 1]$ (convex combination of y and 1). Similarly, $x\tilde{+}y = x + y - xy = y + (1-y)x \in [x, 1]$. Hence $x\tilde{+}y \in [\max(x, y), 1]$, i.e., $\tilde{+}_{s\in S} \in [\sup_{s\in S}, 1]$. The last result is obvious.

Theorem 3 *Let A_{λ_1}, B_{λ_2} are two fuzzy points in Definition 1, $\mu_A(u) = \lambda_1$, $\mu_B(v) = \lambda_2$, then*

$$\mu_{\tilde{d}(A,B)}(\delta) = \begin{cases} 0 & \text{if } \delta \neq d(u,v) \\ \min(\lambda_1, \lambda_2) & \text{otherwise.} \end{cases}$$

It also holds when sup is replaced by $\hat{+}$. If min is replaced by product, then

$$\mu_{\tilde{d}(A,B)}(\delta) = \begin{cases} 0 & \text{if } \delta \neq d(u,v) \\ \lambda_1\lambda_2 & \text{otherwise.} \end{cases}$$

The proof of this theorem is not difficult to obtain by using the above Lemma, and is omitted here.

Theorem 4 *If A and B satisfy the conditions in Definition 3, $\mu_A(u) = \mu_B(v) = 1$, then*

$$\mu_{\tilde{d}(A,B)}(\delta) \begin{cases} = 1 & \text{if } \delta = d(u,v) \\ = 0 & \text{if } \delta > \sup\{d(w,z) :\ w \in S_A, z \in S_B\} \\ \in (0,1) & \text{otherwise,} \end{cases}$$

where S_A and S_B are supports of A and B, respectively. It also holds when sup is replaced by $\hat{+}$ or min is replaced by product. In the special case, when $A = B$, $\mu_{\tilde{d}(A,B)}(0) = 1$.

Proof. If $\delta = d(u,v)$, then

$$\begin{aligned} 1 &= \min(\mu_A(u), \mu_B(v)) \\ &\geq \sup_{d(w,z)=\delta} \min(\mu_A(w), \mu_B(z)) \\ &\geq \min(\mu_A(u), \mu_B(v)) = 1, \end{aligned}$$

we have $\mu_{\tilde{d}(A,B)}(\delta) = 1$. It is obvious that these inequalities also hold when min is replaced by product.

If $\delta > \sup\{d(w,z) : w \in S_A, z \in S_B\}$, then for any w and z satisfying $\delta = d(w,z)$, at least one of $\mu_A(w)$ and $\mu_B(z)$ is zero, so $\mu_{\tilde{d}(A,B)}(\delta) = 0$ no matter if min or product is used.

By noticing Lemma 1, it is not difficult to see that the results are also true when sup is replaced by the probabilistic sum.

By noticing Theorem 2, we get

Corollary 1 *The results in Theorem 4 are also true for points in Definition 2.*

Fuzzy points can be considered as a fuzzy set of $\mathbb{R}$. We illustrate this here by an example. Let us consider the two dimensional normal density function [5] (random variables are somewhat similar to fuzzy sets):

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}} exp\left\{-\frac{\sigma_1^2\sigma_2^2}{2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}\right. \\ &\quad \left.\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\sigma_{12}\frac{(x-\mu_1)^2(y-\mu_2)^2}{\sigma_1^2\sigma_2^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\} \end{aligned}$$

where μ_1, μ_2 are the means, σ_1^2, σ_2^2 are the variances for the random variables $\tilde{x}, \tilde{y}$, respectively, and σ_{12} is the covariance.

Since $\sigma_1 > 0$, $\sigma_2 > 0$, and $\left|\frac{\sigma_{12}}{\sigma_1\sigma_2}\right| < 1$, we have

$$\left(\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2}\right)^2 - \frac{1}{\sigma_1^2\sigma_2^2} < 0,$$

hence

$$\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\sigma_{12}\frac{(x-\mu_1)^2(y-\mu_2)^2}{\sigma_1^2\sigma_2^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] \geq 0 \quad \forall x, y,$$

therefore

$$0 < f(x,y) \leq \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}}.$$

If we let

$$g(x,y) = f(x,y) / \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}},$$

and

$$\mu_A(x,y) = g(x,y),$$

then A is a fuzzy point satisfying Definition 3.

The projection of the intersection of any horizontal plane $z = k$ ($k \in (0,1]$) with $g(x,y)$ onto the xy−plane is an ellipse, all these ellipses are of the same center (μ_1, μ_2), (when $\sigma_1 = \sigma_2$, there are concentrical circles), the membership function remains a constant on any such ellipse. This gives us some hints that this fuzzy point A is no more than a fuzzy set on $\mathbb{R}$.

In fact, if we define the following function

$$\mu(d) = k, \qquad d > 0,$$

where

$$k = exp\left\{-\frac{\sigma_2^2}{2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}d\right\},$$

so $0 < k \leq 1$, then for any $(x, y) \in \mathbb{C}$,

$$g(x, y) = \mu(d),$$

where

$$d = (x - \mu_1)^2 - 2\frac{\sigma_{12}}{\sigma_2^2}(x - \mu_1)^2(y - \mu_2)^2 + \frac{\sigma_1^2}{\sigma_2^2}(y - \mu_2)^2.$$

Therefore we can write

$$\int_{X \times Y} f(x, y)/(x, y) = \int_{\mathbb{R}^+} \mu(d)/d,$$

where

$$\mathbb{R}^+ = \{x : \ x \geq 0\}.$$

The above horizontal-plane-cutting method can be generalized to any fuzzy points.

More generally, we can define a fuzzy point as

Definition 4 *A fuzzy point is a convex fuzzy set A of the plane* $\mathbb{C}$ *such that*

(a) $\exists$ *a convex finite region* $\Omega \in \mathbb{C}$ *satisfying* $\mu_A(\Omega) = \lambda$ *for some* $\lambda \in [0, 1]$,

(b) $\max(\mu_A(\Omega)) = \lambda$,

(c) μ_A *is continuous.*

A method of specifying the membership function of a fuzzy point A satisfying Definition 4 is via the *truncated cone function*; i.e.

$$\mu_A(x, y) \begin{cases} = 0 & \text{if} \quad r_2 < d \\ = \lambda \dfrac{r_2 - d}{r_2 - r_1} & \text{if} \quad r_1 \le d \le r_2 \\ = \lambda & \text{if} \quad d < r_1, \end{cases}$$

where $d = \sqrt{(x - x_p)^2 + (y - y_p)^2}$ = distance from the *centre* (x_p, y_p) of the fuzzy point A. The radii r_1 and r_2 are called the minimum and maximum radius, respectively, of the truncated cone, and $r_2 > r_1$.

Figure 1 is a plot of this function versus distance d. Figure 2 shows a three dimensional plot of $\mu_A(x, y)$ for two points, one with $(x_p, y_p) = (3, 6)$, and $r_1 = 1.0$, $r_2 = 2.0$, and the second one with $(x_p, y_p) = (8, 7)$, and $r_1 = 0.5$, $r_2 = 1.5$. For both points, $\lambda = 1$.

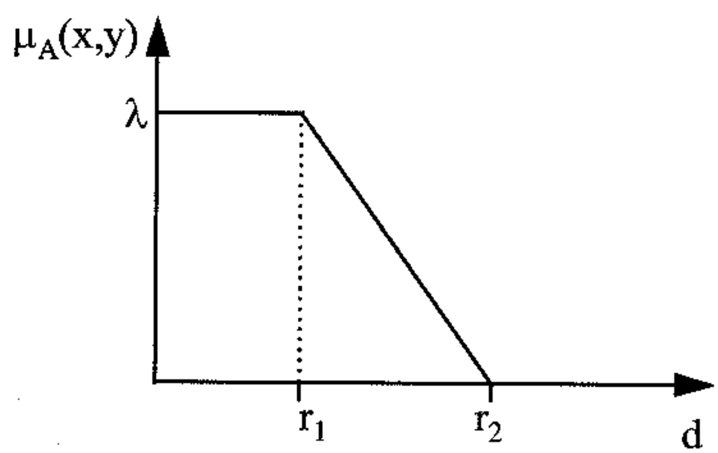

Figure 1. Truncated cone function.

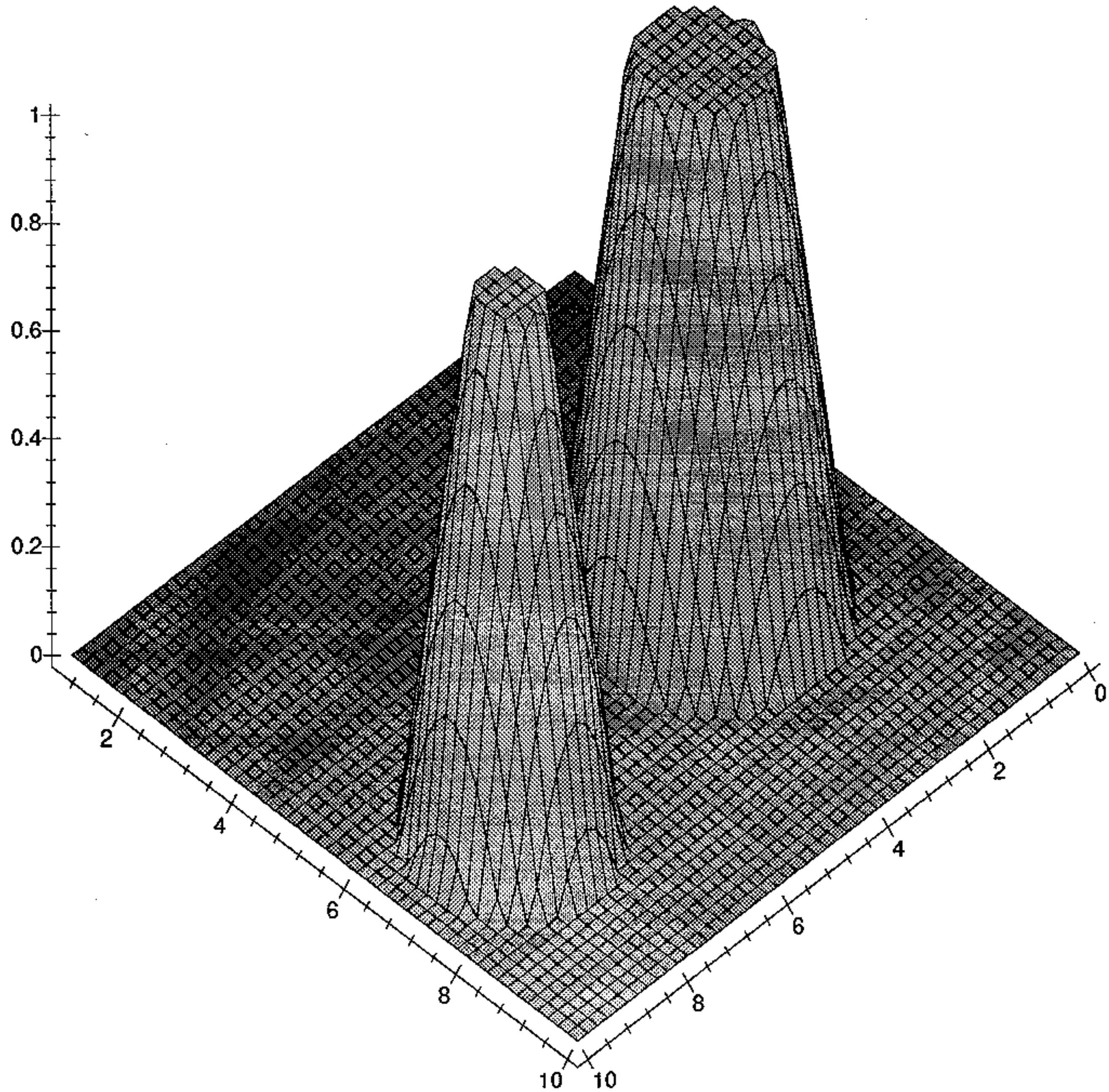

Figure 2. The membership functions of two fuzzy points.

3 Geometric Properties

In this section, we will use the sup-min idea of extension principle to introduce some geometrical properties on $\tilde{\mathbb{C}}$. *Fuzzy line, fuzzy line segment, fuzzy angle, fuzzy perpendicular, fuzzy parallel, fuzzy circle and fuzzy area of fuzzy polygon* are introduced. First, we discuss how to determine a fuzzy line after two different fuzzy points are given.

A *fuzzy line* determined by two fuzzy points A and B, $\tilde{l}(A, B)$, is defined as a fuzzy set on $\tilde{\mathbb{C}}$:

$$\mu_{\tilde{l}(A,B)}(C) = \sup_{w \in l(u,v)} min(\mu_A(u), \mu_B(v), \mu_C(w)),$$

where $w \in l(u, v)$ means that w is on the crisp line passing through u and v.

A *fuzzy line segment* between two fuzzy points A and B can be expressed as the set of fuzzy points: $[A, B] = \{\alpha A + (1 - \alpha)B : \quad \alpha \in [0, 1]\}$, where the membership function

$$\mu_{\alpha A+(1-\alpha)B} = \alpha\mu_A + (1 - \alpha)\mu_B.$$

Figure 3 is an illustration of the membership function of the fuzzy line segment between the two fuzzy points illustrated in Figure 2.

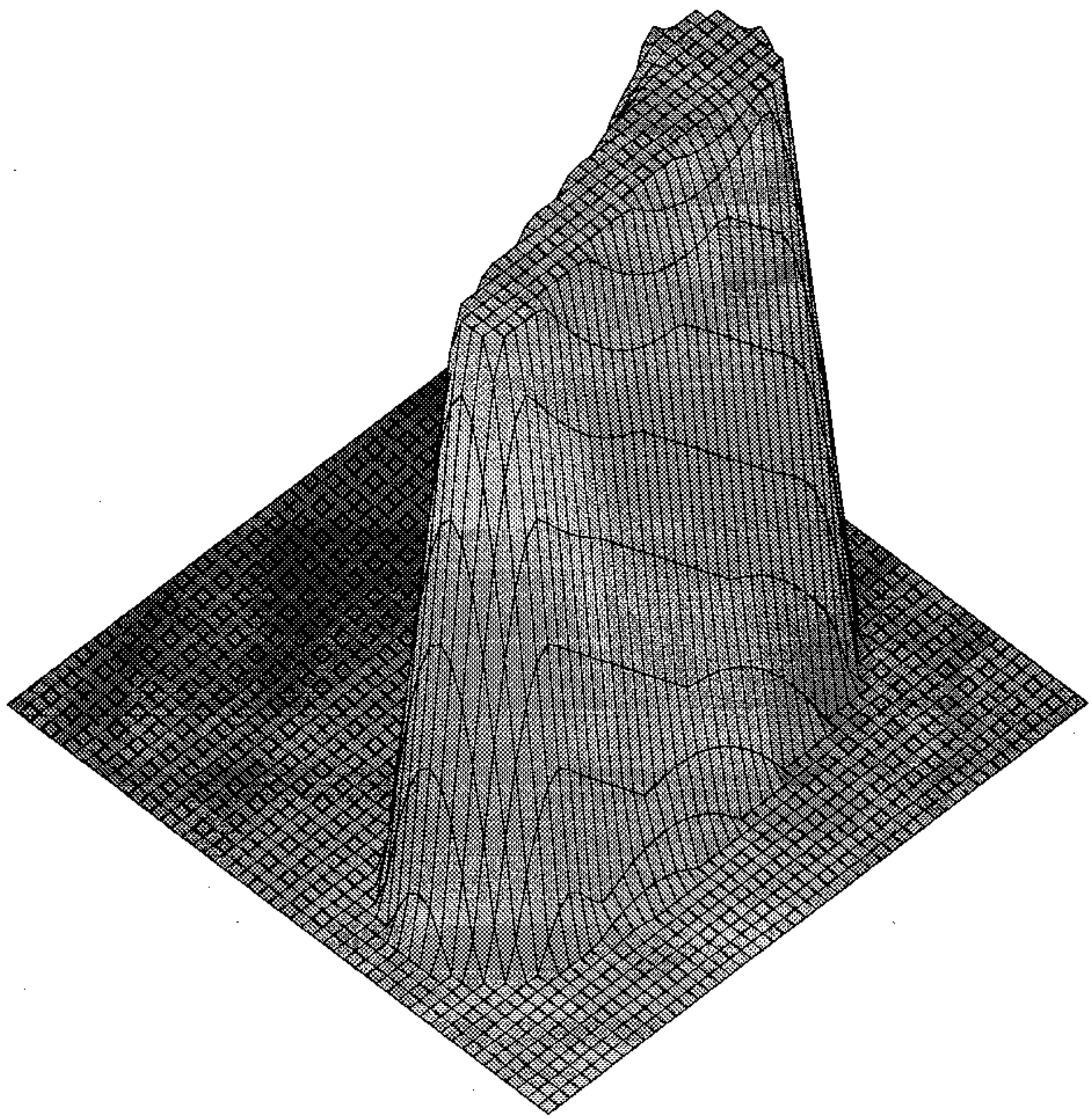

Figure 3. The membership function of a fuzzy line segment.

If A, B and C are three fuzzy points defined by Definition 1, $\mu_A(u) = \lambda_1$, $\mu_B(v) = \lambda_2$, $\mu_C(w) = \lambda_3$, then

$$\mu_{\tilde{l}(A,B)}(C) = \begin{cases} \min(\lambda_1, \lambda_2, \lambda_3) & \text{if } w \in l(u,v) \\ 0 & \text{otherwise} \end{cases}$$

For a fuzzy line $\tilde{l}(A,B)$, we can define its *fuzzy slope*, $\tilde{s}(A,B)$, as follows

$$\mu_{\tilde{s}(A,B)}(m) = \sup_{m=s(u,v)} \min(\mu_A(u), \mu_B(v)), \quad m \in \mathbf{R},$$

where $s(u,v)$ is the slope of the crisp line pass through u and v. Fuzzy slope is a fuzzy set on $\mathbf{R}$.

If A, B are two fuzzy points defined by Definition 1, $\mu_A(u) = \lambda_1$, $\mu_B(v) = \lambda_2$, then

$$\mu_{\tilde{s}(A,B)}(m) = \begin{cases} \min(\lambda_1, \lambda_2) & \text{if } m = s(u,v) \\ 0 & \text{otherwise} \end{cases}$$

After we have fuzzy line, we can introduce the concept of fuzzy angle formed by two fuzzy lines. The *fuzzy angle* from $\tilde{l}(A,B)$ to $\tilde{l}(C,D)$, $< \tilde{l}(A,B)\tilde{l}(C,D)$, can be defined as

$$\forall \alpha \in [0, 2\pi),$$

$$\mu_{<\tilde{l}(A,B)\tilde{l}(C,D)}(\alpha) = \sup_{\alpha=<\tilde{l}(u,v)\tilde{l}(w,z)} \min(\mu_A(u), \mu_B(v), \mu_C(w), \mu_D(z)),$$

where $< \tilde{l}(u,v)\tilde{l}(w,z)$ is the angle from the line $l(u,v)$ to $l(w,z)$.

Given a fuzzy line $\tilde{l}(A,B)$, we can define how it is *perpendicular* to other fuzzy lines by

$$\mu_{\tilde{l}(A,B)\perp}(\tilde{l}(C,D)) = \sup_{l(u,v)\perp l(w,z)} \min(\mu_A(u), \mu_B(v), \mu_C(w), \mu_D(z)).$$

From the definition, if we have four crisp points A, B, C, D, then

$$\mu_{\tilde{l}(A,B)\perp}(\tilde{l}(C,D)) = \begin{cases} 1 & \text{if } l(u,v) \perp l(w,z) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can introduce in what degree two fuzzy lines are *parallel* to each other

$$\mu_{\tilde{l}(A,B)\|}(\tilde{l}(C,D)) = \sup_{l(u,v)\| l(w,z)} \min(\mu_A(u), \mu_B(v), \mu_C(w), \mu_D(z)).$$

From the definition, if we have four crisp points A, B, C, D, then

$$\mu_{\tilde{l}(A,B)\|}(\tilde{l}(C,D)) = \begin{cases} 1 & \text{if } l(u,v) \parallel l(w,z) \\ 0 & \text{otherwise.} \end{cases}$$

We now would like to introduce the concept of *fuzzy circle*, $\tilde{O}(A,r)$, with a fuzzy point A as its center and a positive real number r as its radius:

$$\mu_{\tilde{O}(A,r)}(C) = \sup_{d(u,v)=r} \min(\mu_A(u), \mu_C(v)),$$

where C is a fuzzy point. $\tilde{O}(A, r)$ is a fuzzy set on $\tilde{\mathbb{C}}$.

If r is also a fuzzy number, then we define $\tilde{O}(A, r)$ as

$$\mu_{\tilde{O}(A,r)}(C) = \sup_{d(u,v)=t} \min(\mu_A(u), \mu_C(v), \mu_r(t)),$$

Consider a *fuzzy polygon* $E_n = [A_1, A_2, \ldots, A_n]$ formed by n fuzzy line segments $[A_1, A_2]$, $[A_2, A_3]$, ..., $[A_n, A_1]$, we can define its *fuzzy area, $area(E_n)$*, by

$$\forall \alpha \in \mathbb{R}^+, \quad \mu_{area(E_n)}(\alpha) = \sup_{\alpha = area([a_1,\ldots,a_n])} \min_{1 \le i \le n} \mu_{A_i}(a_i),$$

where $area([a_1, \ldots, a_n])$ means the area of the crisp polygon with vertexes $\{a_1, \ldots, a_n\}$.

The definition here is a generalization of area. If the fuzzy points A_i satisfy $\mu_{A_i}(u_i) = 1$, $i = 1, 2, \ldots, n$, and $a = area([u_1, u_2, \ldots, u_n])$, then $\mu_{area(E_n)}(a) = 1$.

4 Analytic and Topological Properties

In this section, we would like to introduce *fuzzy limits of a sequence of fuzzy points, limits of sequence of fuzzy points, closed, open, and compact subsets of* $\tilde{\mathbb{C}}$, by using these concepts, we discuss some approximation properties in

$\tilde{\mathbb{C}}$.

The *fuzzy limits of a sequence of fuzzy points* A_n is defined as a fuzzy subset on $\tilde{\mathbb{C}}$:

$$\mu_{\lim\limits_{n\to\infty}(A_n)}(A) = \sup_{\lim\limits_{n\to\infty}(u_n)=u} \sup_{k\geq 1} \min_{n\geq k}(\mu_{A_n}(u_n), \mu_A(u)).$$

If $A_n = \lambda_n/u_n,\ A = \lambda/u$, then

$$\mu_{\lim\limits_{n\to\infty}(A_n)}(A) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} u_n \neq u \\ \sup_{k\geq 1}\min_{n\geq k}(\lambda_n, \lambda) & \text{otherwise.} \end{cases}$$

In special case, if $\lim_{n\to\infty} u_n = u$, and $\lim_{n\to\infty} \lambda_n = \lambda$, then $\forall \epsilon > 0$, $\exists K$, such that $|\lambda_n - \lambda| < \epsilon$, (i.e., $\lambda - \epsilon < \lambda_n < \lambda + \epsilon$), for any $n \geq K$. This implies

$$\sup_{k\geq 1}\min_{n\geq k}(\lambda_n, \lambda) = \lim_{n\to\infty} \lambda_n = \lambda.$$

A fuzzy point is uniquely determined by its characteristic function, so we can consider a fuzzy point as a 2-dimensional function, and extend some properties of functions to fuzzy points.

Definition 5 *Suppose* f *is a function on* $\mathbb{C}$*, its norm, expressed by* $\|f\|$ *is defined by*

$$\|f\| = \sup_{c\in\mathbb{C}} |f(c)|.$$

The *limit of a sequence of fuzzy points* is defined by

$$\lim_{n\to\infty} A_n = A \text{ iff } \lim_{n\to\infty} \|\mu_{A_n} - \mu_A\| = 0.$$

From this definition, we know if $A_n = \lambda_n/u_n,\ A = \lambda/u$, then

$$\lim_{n\to\infty} A_n = A \text{ iff } \lim_{n\to\infty} \lambda_n = \lambda \text{ and } \lim_{n\to\infty} u_n = u.$$

With the limit concept, we can introduce the following definition

Definition 6 *(a) A subset $E \subset \tilde{\mathbb{C}}$ is called closed, if $\{B_n\} \subset E$, $B_n \longrightarrow B$ implies $B \in E$.*

(b) the complement, E^c, of a subset $E \subset \tilde{\mathbb{C}}$ is defined by

$$\mu_{E^c} = 1 - \mu_E.$$

(c) A subset $E \subset \tilde{\mathbb{C}}$ is called open if E^c is closed.

(d) A subset $E \subset \tilde{\mathbb{C}}$ is called compact, if $\forall\{B_n\} \subset E$, $\exists\{B_{n_k}\} \subset \{B_n\}$, such that $B_{n_k} \longrightarrow B_0 \in E$.

From the definition, we can see

Corollary 2 *A compact subset is closed.*

Now we want to contribute some approximation properties to this project.

Definition 7 *Let $E \subset \tilde{\mathrm{C}}$, $A_0 \in \tilde{\mathrm{C}}$. If there exists $B_0 \in E$, such that for any $B \in E$, we have*

$$\|\mu_{A_0} - \mu_{B_0}\| \leq \|\mu_{A_0} - \mu_B\|,$$

then we call B_0 a best approximant to A_0 from E.

Theorem 5 *If E is a compact subset of $\tilde{\mathrm{C}}$, then $\forall A_0 \in \tilde{\mathrm{C}}$, $\exists B_0 \in E$, such that B_0 is a best approximant to A_0 from E.*

Proof. Let $\{B_n\} \subset E$ satisfy

$$\lim_{n\to\infty} \|\mu_{B_n} - \mu_{A_0}\| = \inf_{B\in E} \|\mu_B - \mu_{A_0}\|,$$

since E is compact, by definition, there exists a subsequent $\{B_{n_k}\} \subset \{B_n\}$, such that $\lim_{n\to\infty} B_{n_k} = B_0 \in E$, hence

$$\|\mu_{B_0} - \mu_{A_0}\| = \lim_{k\to\infty} \|\mu_{B_{n_k}} - \mu_{A_0}\| = \inf_{B\in E} \|\mu_B - \mu_{A_0}\|,$$

i.e., B_0 is a best approximant to A_0 from E.

Theorem 6 *Let $E \subset \tilde{\mathrm{C}}$, $A_0 \in \tilde{\mathrm{C}}$. If $B_0 \in E$ is a best approximant to A_0, then B_0 is also a best approximant to any point on the fuzzy line segment $[A_0, B_0]$.*

Proof. If B_0 is a best approximant to A_0, $\forall\lambda \in [0,1]$, $\forall B \in E$, then

$$\begin{aligned}
& \|\mu_{\lambda A_0+(1-\lambda)B_0} - \mu_B\| \\
= & \|\lambda\mu_{A_0} + (1-\lambda)\mu_{B_0} - \mu_B\| \\
= & \|\lambda\mu_{A_0} + (1-\lambda)\mu_{B_0} - \mu_{A_0} + \mu_{A_0} - \mu_B\| \\
\geq & \|\mu_{A_0} - \mu_B\| - \|\lambda\mu_{A_0} + (1-\lambda)\mu_{B_0} - \mu_{A_0}\| \\
\geq & \|\mu_{A_0} - \mu_{B_0}\| - \|\lambda\mu_{A_0} + (1-\lambda)\mu_{B_0} - \mu_{A_0}\| \\
= & \|\mu_{A_0} - \mu_{B_0}\| - (1-\lambda)\|\mu_{A_0} - \mu_{B_0}\| \\
= & \lambda\|\mu_{A_0} - \mu_{B_0}\|.
\end{aligned}$$

That is

$$\inf_{B\in E} \|\mu_{\lambda A_0+(1-\lambda)B_0} - \mu_B\| \geq \lambda\|\mu_{A_0} - \mu_{B_0}\|.$$

But

$$\begin{aligned}
\|\mu_{\lambda A_0+(1-\lambda)B_0} - \mu_{B_0}\| &= \|\lambda\mu_{A_0} + (1-\lambda)\mu_{B_0} - \mu_{B_0}\| \\
&= \lambda\|\mu_{A_0} - \mu_{B_0}\|.
\end{aligned}$$

Therefore B_0 is also a best approximant to $\lambda A_0 + (1-\lambda)B_0$. The proof is completed.

Definition 8 *Let E_1, E_2 be two subset of* $\mathbb{C}$, *If there exists $B_0 \in E_2$, such* *that*

$$\sup_{A \in E_1} \|\mu_A - \mu_{B_0}\| = \inf_{B \in E_2} \sup_{A \in E_1} \|\mu_A - \mu_B\|,$$

then we call B_0 a best simultaneous approximant to E_1 from E_2.

Similar to the best approximant case, we can prove the following theorem

Theorem 7 *If E_2 is compact, then for any nonempty subset $E_1 \subset \tilde{\mathbb{C}}$, the best simultaneous approximants to E_1 from E_2 exist.*

The proof is similar to that of Theorem 5, and is omitted here.

References

[1] Dubois, D. and Prade, H., *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980, pp. 36 - 40.

[2] Dubois, D. and Prade, H., "Comment on "Tolerance analysis using fuzzy sets" and "A procedure for multiple aspect decision making."", *Int. J. Syst. Sci.*, **9**, No. 3, 1978, pp. 357 - 360.

[3] Gaines, B. R., "Foundations of fuzzy reasoning", *Int. J. of Man-Machine Studies*, **8**, 1976, pp. 623 - 668.

[4] Jain, R., "Tolerance analysis using fuzzy sets", *Int. J. Syst. Sci.*, **7**, No. 12, 1976, pp. 1392 - 1401.

[5] Mikhail, E. M., *Observations and Least Squares*, IEP - A Dun-Donnelley Publisher, NY, 1976.

[6] Zadeh, L. A., "The concept of a linguistic variable and its applications to approximate reasoning. Parts 1, 2 and 3", *Information Sciences*, **8**, 1975, pp.199-249, 301-357, **9**, pp.43-80.